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Binary Choice Probabilities on Mixture Sets

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Abstract

Experimental evidence suggests that choice behaviour has a stochastic element. Much of this evidence is based on studying choices between lotteries – choice under risk. Binary choice probabilities admit a *strong utility representation (SUR)* if there is a utility function such that the probability of choosing option A over option B is a strictly increasing function of the utility difference between A and B. Debreu (1958) obtained a simple set of sufficient conditions on binary choice probabilities for the existence of a SUR. More recently, Dagsvik (2008) considered binary choices between lotteries and provided axiomatic foundations for a SUR in which the underlying utility function is linear (i.e., conforms to expected utility). Our paper strengthens and generalises Dagsvik's result. We show that one of Dagsvik's axioms can be weakened, and we extend his analysis to encompass choices between uncertain prospects, as well as various non-linear specifications of utility.

1 Introduction

Experimental evidence on risky choice is conventionally interpreted as robustly rejecting the descriptive validity of expected utility (EU) theory. This evidence (or rather, its interpretation) has spawned a range of alternatives to EU.

Less well known, perhaps, is the fact that experiments also provide robust evidence of randomness in choice behaviour – subjects are often observed to make different choices in successive presentations of the same choice problem. Since the early 1990s increasing attention has been paid to this latter phenomenon, and to the way in which "noise" is modelled in the analysis of experimental data on choice behaviour. This new line of inquiry has prompted some revisionist thinking about the descriptive merits of EU theory. Indeed, by 1995 John Hey was prepared to advance the following tentative hypothesis:

> "[O]ne can explain experimental anlayses of decision making under risk better (and simpler) as EU plus noise – rather than through some higher level functional – as long as one specifies the noise appropriately." (Hey, 1995, p.640)

In support of Hey's hypothesis, consider the evidence of Schmidt and Neugebauer (2007). They presented 24 subjects with the same 28 binary choice problems on three consecutive days. These 28 problems were constructed so as to create ample opportunity for the exhibition of Allais-type violations of independence: the set of binary choices contains multiple (28) instances of common consequence relationships, common ratio relationships or a mixture of the two. For a given pair of choice problems, Schmidt and Neugebauer called a subject a "repetition consistent" for that pair if, for each problem in the pair, s/he made the same choice on all three presentations of the problem. Otherwise, the subject was deemed "repetition inconsitent". Schmidt and Neugebauer observe significant rates of repetition inconsistency, even though subjects are allowed to report indifference. More importantly, they find that the majority of independence violations are committed by the "repetition inconsistent" choosers.

This evidence is suggestive of noisy choice behaviour, but of choice behaviour that is guided by expected utility nonetheless.[1] Other studies (such as Buschena and Zilberman, 2000) lend further weight to this suggestion, but contrary evidence can also be found (such as that in Loomes and Sugden, 1998, and Loomes and Pogrebna, 2014).

This new evidence not only challenges experimentalists to revisit the empirical analysis of their data; it also issues a challenge to decision theorists. It has revived interest (amongst economists, that is – psychologists have maintained a continuous level of interest) in models of stochastic choice. These models characterise decision-mkaers in terms of choice probabilities, rather than preference relations. The theoretical challenge is to devise parsimonious, but descriptively accurate, representations for these choice probabilities validated by plausible sets of axiomatic restrictions.

For *binary* choice problems, the class of Fechner representations are commonly used. A Fechner model treats choice as fundamentally random, with choice probabilities determined by the difference in the utility-stimulus of each option. A Fechner model thus

[1] In a similar spirit, Blavatskyy (2006) revisits the experimental evidence against the betweenness property of preferences. He finds that this evidence is also compatible with noisy, but betweenness-satisfying, choice behaviour.

involves a utility function that determines the strength of stimuli and an auxiliary function that converts utility differences into the probability of choosing the utility-maximising alternative.

In this paper we focus on a particular type of Fechner model, sometimes called a *strong utility* model, in which this auxiliary function is required to be *strictly increasing* – the greater the utility difference, the higher the probability of making a utility-maximising choice.

In the context of binary choices between *lotteries*, it is natural to ask whether the data are consistent with a strong utility model whose utility function is drawn from a particular class, such those with the expected utility form. This class is sometimes called the *core theory* of the model. For a given core theory (i.e., class of utility functions), one seeks necessary and sufficient conditions on binary choice probabilities for a strong utility representation with respect to a utility function drawn from this class. The necessary conditions provide testable restrictions on observed probabilities; the set of sufficient conditions provides some basis for assessing the theoretical plausibility of the model.

At present, there is limited understanding of the axiomatic foundations of strong utility models for choice under risk or uncertainty. Dagsvik (2008) is an important recent contribution for the case of choice between lotteries, Dagsvik provides a set of sufficient conditions for a *strong expected utility representation (SEUR)* – that is, a strong utility representation with EU as the core theory. To the best of our knowledge, no axiomatic foundations have been provided for strong utility models (in the context of risk) with core theories other than EU, or for strong utility models for choice between uncertain (as opposed to risky) prospects. The present paper fills some of this gap. It also proves that Dagsvik's (2008) set of sufficient conditions for a SEUR may be weakened.

1.1 Related literature

As noted, our results build directly on the work of Dagsvik (2008), which is discussed in more detail below. Here, we briefly mention the related work of Blavattsky (2008, 2012) and Gul and Pesendorfer (2006).

In the context of risk, Blavatskyy (2008) axiomatises a Fechner model – with EU as the core theory – but not of the strong utility form. In Blavatskyy's representation, the function that converts utility differences into choice probabilities is required to be non-decreasing but need not be strictly increasing. Choice probabilities can therefore be constant over some ranges of utility differences.[2] In terms of axioms, the critical difference between Dagsvik (and the present paper) and Blavatskyy is in the stochastic version of the independence axiom that is used: *strong independence* (Dagsvik, 2008, and here) versus *common consequence independence* (Blavatskyy, 2008).[3]

In the same paper, Blavatskyy also proposes a modification of his axioms to accommodate implicit EU (Dekel, 1986) as the core theory. Such utility functions have contours that are linear but not necessarily parallel – they satisfy betweenness but not necessarily independence. To the best of our knowledge, this is the only existing axiomatisation of a Fechner model for lotteries in which the core theory is other than EU. The present paper provides one more: we axiomatise a strong utility model with the Yaari's (1987) Dual Theory at its core.[4]

For the representation of binary choice probabilities, the main rivals to the Fechner models are the *random utility* models.[5] In a random utility model, the decision-maker has a *set* of utility functions, one of which is randomly drawn – according to a fixed probability measure – whenever the decision-maker is confronted with a choice problem. The randomly selected utility function is then maximised without error. Gul and Pesendorfer (2006) axiomatise a random *expected utility* model, which requires that all utility functions in the set have the EU form. We are not aware of any work on random utility models with non-EU functions or for choice under uncertainty.

Blavatskyy (2012) considers binary choices between uncertain prospects (Savage acts). The model that Blavatskyy axiomatises is a close relative of the Fechner models, though not, strictly speaking, within the Fechner class. Instead, the probability of making a utility-maximising choice depends on the "normalised" utility difference. Specifically, given two uncertain prospects, their utility difference is normalised by the difference between the utility of the *mutually dominating prospect* that gives the *best* of the outcomes provided by the two prospects in every state, and that of the *mutually dominated prospect* that gives the *worst* of the two outcomes in every state. Blavatskyy axiomatises such a model when the core theory is subjective expected utility (SEU).

So far as we are aware, this is the only existing axiomatisation of a Fechner-like model

[2]It is easy to check that Example 1 satisfies all of Blavatskyy's axioms when $\hat{u}$ is linear.

[3]However, while our axioms and representations are in the spirit of Dagsvik (2008), the structure of our proofs is closer to that in Blavatskyy (2008).

[4]Unfortunately, we have not been able to axiomatise the strong utility model for the wider class of rank-dependent expected utility functions.

[5]A very strong rival according to the results of Loomes, Moffatt and Sugden (2002) and Butler, Isoni and Loomes (2012).

for choice between uncertain prospects. Once again, the present paper is complementary. We consider Anscome-Aumann acts, rather than Savage acts. We obtain representations of the conventional strong utility form for three alternative core theories: SEU, maxmin expected utility and Choquet expected utility.

It should be noted that a major limitation of the Fechner models is their failure to accommodate evidence that decision-makers rarely choose dominated options when there is a transparent dominance relationship between the alternatives – either first-order stochastic dominance (risk) or statewise dominance (uncertainty).[6] Even when dominance relationships are not present, recent empirical work casts doubt on the descriptive adequacy of the strong utility structure.[7] However, while acknowledging the empirical limitations of the Fechner models, given their "benchmark" status in the literature it is still useful to understand their axiomatic foundations.

2 Binary stochastic choice

Debreu (1958) proved a famous representation theorem for *binary stochastic choice*. The primitives of the theory are a pair (A, P), where A is an arbitrary set of alternatives and P is a *binary choice probability function (BCPF)*. The latter is a mapping

$$P : A \times A \to [0, 1]$$

satisfying

$$P(a, b) = 1 - P(b, a) \tag{1}$$

for all $a, b \in A$.

The quantity $P(a, b)$ is the probability with which the decision-maker selects a when given the choice between a or b (with abstention not being an option). This interpretation is behaviourally meaningful only if $a \neq b$, but it is traditional to define P on the entire Cartesian product $A \times A$ for convenience. An immediate implication of (1) is that

$$P(a, a) = \frac{1}{2}$$

for all $a \in A$.

[6]See, for example, Loomes, Moffatt and Sugden (2002). Blavatskyy (2012) uses a (non-Fechnerian) representation with normalised utility differences precisely to avoid this problem. For the same reason, Blavatskyy (2011) introduces a suitably normalised model for choice under risk.

[7]See Butler, Isoni and Loomes (2012).

Given P, it is natural to impute the following binary relation on A: for any $a, b \in A$:

$$a \succsim^P b \quad \Leftrightarrow \quad P(a,b) \geq P(b,a) \quad \Leftrightarrow \quad P(a,b) \geq \frac{1}{2} \tag{2}$$

where the second equivalence follows from (1). In other words, a is "weakly preferred" to b iff the decision-maker chooses a over b at least half of the time. The asymmetric and symmetric parts of $\succsim^P$ are denoted $\succ^P$ and $\sim^P$ respectively, and satisfy

$$a \succ^P b \quad \Leftrightarrow \quad P(a,b) > \frac{1}{2}$$

and

$$a \sim^P b \quad \Leftrightarrow \quad P(a,b) = \frac{1}{2}.$$

Debreu provides sufficient conditions for the existence of a utility function $u : A \to \mathbb{R}$ such that

$$P(a,b) \geq P(c,d) \quad \text{iff} \quad u(a) - u(b) \geq u(c) - u(d) \tag{3}$$

Choice probabilities are thus determined by utility differences, in the Fechnerian tradition of psychophysics (Falmagne, 2002).[8] Following Marschak (1960) we say that P has a *strong utility representation (SUR)* when (3) obtains for some u.[9]

Debreu's sufficient conditions comprise two axioms, the first of which is the *quadruple condition* (Davidson and Marschak, 1959):

Axiom D1 For all $a, b, a', b' \in A$:

$$P(a,b) \geq P(a',b') \quad \text{iff} \quad P(a,a') \geq P(b,b')$$

The quadruple condition is a close relative of the following:

Strong Stochastic Transitivity (SST) For all $a, b, c \in A$, if

$$\min\{P(a,b),\ P(b,c)\} \ \geq\ \frac{1}{2}$$

then

$$P(a,c) \ \geq\ \max\{P(a,b),\ P(b,c)\}.$$

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[8] There is a closely related literature in which the primitive P is replaced by a weak order $\hat{\succsim}$ on $A \times A$, called a *difference relation*, and sufficient conditions are sought for a *utility-difference* representation:

$$(a,b) \,\hat{\succsim}\, (c,d) \quad \Leftrightarrow \quad u(a) - u(b) \geq u(c) - u(d).$$

See Köbberling (2006) for a recent example and a summary of the previous literature.

[9] The literature is not consistent in the use of terminology. Some authors (e.g., Luce and Suppes, 1965) use "strong utility" for a representation in which (3) is only required to hold when $P(a,b) \notin \{0,1\}$ and $P(c,d) \notin \{0,1\}$.

It is well known that Axiom D1 implies SST, but not conversely.[10] Davidson and Marschak (1959, p.240) prove that SST is equivalent to the following *weak substitutability* condition:[11] for any $a, b, c \in A$,

$$P(a,b) \geq \frac{1}{2} \quad \Rightarrow \quad P(a,c) \geq P(b,c) \tag{4}$$

The extra strength of Axiom D1 comes at the cost of some intuitive appeal. The SST condition (or weak substitutability) has a familiar and transparent logic, while the quadruple condition is arguably less compelling from a normative point of view.

The second of Debreu's (1958) axioms is a *solvability* condition:

Axiom D2 For all $a, b, c \in A$ and all $\pi \in (0,1)$

$$P(a,b) \geq \pi \geq P(a,c) \quad \Rightarrow \quad P(a,e) = \pi \text{ for some } e \in A$$

Debreu proves that Axioms D1–D2 suffice for a strong utility representation.[12]

It is important to note that Axiom D1 cannot be replaced by the weaker SST without jeopardising Debreu's result.[13]

Example 1. Let A be a compact, convex subset of some Euclidean space and let $\hat{u}$: $A \to \mathbb{R}$ be a continuous function with $\hat{u}(A) = [\underline{u}, \overline{u}] \subseteq [0,1]$ and $\left[\frac{1}{4}, \frac{3}{4}\right] \subseteq (\underline{u}, \overline{u})$.

[10] See Luce and Suppes (1965, Theorem 39). Example 1, described below, also shows that SST does not imply the quadruple condition.

[11] If we replace "$\Rightarrow$" with "$\Leftrightarrow$" in (4) we obtain the *substitutability* condition (Tversky and Russo, 1969), which is stronger (as Example 1 below illustrates).

[12] Scott (1964) showed that D1 alone is not sufficient for a SUR, though it is clearly necessary. Axiom D2 is not necessary, as the following example illustrates: $A = [0,1]$ and

$$P(a,b) = \begin{cases} \frac{3}{4} + \frac{1}{4}(a-b) & \text{if } a > b \\ \frac{1}{2} & \text{if } a = b \\ \frac{1}{4} + \frac{1}{4}(a-b) & \text{if } a < b \end{cases}$$

Then $P(a,b) \geq P(c,d)$ iff $a - b \geq c - d$ but solvability is violated: for example, take any a, b, c with $c > a > b$ and any $\pi \in \left(\frac{1}{4}, \frac{1}{2}\right)$.

[13] Köbberling (2006, Theorem 1) shows that the QC, which is called the *strong crossover* property when translated into the language of difference relations, can be weakened within her axiom system while preserving the SUR. However, one of her other axioms, *weak separability*, already implies the substitutability property, which is stronger than SST.

Now define $P : A \times A \to [0,1]$ as follows:

$$P(a,b) = \begin{cases} \frac{3}{4} & \text{if } \frac{1}{2}[1+\hat{u}(a)-\hat{u}(b)] \geq \frac{3}{4} \\ \frac{1}{4} & \text{if } \frac{1}{2}[1+\hat{u}(a)-\hat{u}(b)] \leq \frac{1}{4} \\ \frac{1}{2}[1+\hat{u}(a)-\hat{u}(b)] & \text{otherwise} \end{cases}$$

It is straightforward to check that P satisfies (1) and Axiom D2. It also satisfies weak substitutability (and hence SST):

$$\begin{aligned} P(a,b) \geq \frac{1}{2} \quad & \Leftrightarrow \quad \frac{1}{2}[1+\hat{u}(a)-\hat{u}(b)] \geq \frac{1}{2} \\ & \Leftrightarrow \quad \hat{u}(a) \geq \hat{u}(b) \\ & \Leftrightarrow \quad \frac{1}{2}[1+\hat{u}(a)-\hat{u}(c)] \geq \frac{1}{2}[1+\hat{u}(b)-\hat{u}(c)] \\ & \Rightarrow \quad P(a,c) \geq P(b,c). \end{aligned}$$

However, P does not satisfy Axiom D1. Let $a, b, c \in A$ be such that $\hat{u}(a) > \hat{u}(b) > \frac{3}{4}$ and $\hat{u}(c) = \frac{1}{4}$. Then

$$P(a,b) = \frac{1}{2}[1+\hat{u}(a)-\hat{u}(b)] > \frac{1}{2} = P(c,c)$$

but $P(a,c) = P(b,c) = \frac{3}{4}$. Since Axiom D1 is clearly a necessary condition for a representation of the form (3), no such representation exists for this example.

One consequence of Axiom D2 is that A must be suitably rich. A familiar context that satisfies this richness condition is choice between risky prospects. For example, A may be the unit simplex in $\mathbb{R}^n$, interpreted as the set of lotteries over a fixed outcome set $X = \{x_1, ..., x_n\}$. Dagsvik (2008) proves a strong utility representation theorem for this context. More precisely, he provides sufficient conditions for a SUR in which the utility function $u : A \to \mathbb{R}$ is *linear*. Let us call this a *strong expected utility representation (SEUR)*.[14] Dagsvik's sufficient conditions for a SEUR augment Debreu's axioms with two more, which we discuss below (Section 4.1).

[14]Luce and Suppes (1965, p.360) use the same appellation to refer to a representation that obeys (3) whenever $P(a,b) \notin \{0,1\}$ and $P(c,d) \notin \{0,1\}$, and u is of the expected utility form.

Dagsvik's result is an important refinement of Debreu's theorem for binary choices between risky prospects. The purpose of the present paper is to provide further such refinements, encompassing various non-expected utility models of choice under risk or uncertainty. We also show that Dagsvik's sufficient conditions for a SEUR may be weakened: the quadruple condition can be replaced by SST.

3 Mixture sets

Our central representation result (Corollary 0) requires only that A is a *mixture set* (Herstein and Milnor, 1953).[15] Given $a, b \in A$ and $\lambda \in [0, 1]$, we write $a\lambda b$ for the λ-mixture of a and b. In particular, $a1b = a$ and $a0b = b$. For example, if $a, b \in A \subseteq \mathbb{R}^n$ then

$$a\lambda b = \lambda a + (1 - \lambda)\, b$$

under the standard mixture operation on $\mathbb{R}^n$.

There are many axiom systems for preferences over mixture sets, leading to various classes of utility functions. The best known, of course, are the axiom systems for *expected utility* and the associated class of *(mixture-)linear* utility functions, but a wide range of non-expected utility classes have also been axiomatised for specific mixture-set contexts. It is therefore natural to seek probabilistic extensions of these expected and non-expected utility models, leading to strong utility representations in which the utility function $u : A \to \mathbb{R}$ falls within the relevant class.

To determine suitable classes of utility functions for this purpose, let us introduce the following concept:

Definition 1 Given some $M \subseteq A$ we say that $u : A \to \mathbb{R}$ is *M-linear* if $u(M) = u(A)$ and

$$u(a\lambda b) = \lambda u(a) + (1 - \lambda)\, u(b)$$

for any $a \in A$, any $b \in M$ and any $\lambda \in [0, 1]$.

If $M = A$ then M-linearity is (mixture-)linearity *simpliciter*. However, if M is a proper subset of A, then M-linearity is a weaker property of u. Importantly, several non-EU models (for preferences over A) have M-linear representations for some $M \subseteq A$ (see Section 4).

[15] While mixture sets can be discrete, the richness condition implied by solvability will implicitly exclude such sets.

Given $M \subseteq A$, if P has a strong utility representation with respect to an M-linear utility function u we say that P has a *strong M-linear utility representation.* The following is clearly necessary for such a representation to exist:

Strong M-Independence For any $a, b, c, d \in A$, any $e \in M$ and any $\lambda \in (0, 1)$,

$$P(a, b) \geq P(c, d) \quad \Rightarrow \quad P(a\lambda e, b\lambda e) \geq P(c\lambda e, d\lambda e).$$

Note that Strong M-Independence implies Strong M'-Independence for any $M' \subseteq M$. Strong M-Independence is a weakening of Dagsvik's (2008) Strong Independence axiom:

Strong Independence For all $a, b, a', b', c \in A$ and all $\lambda \in (0, 1)$

$$P(a, b) \geq P(a', b') \quad \Rightarrow \quad P(a\lambda c, b\lambda c) \geq P(a'\lambda c, b'\lambda c).$$

Strong Independence implies Strong M-Independence for any $M \subseteq A$.

Our main representation result is obtained as a corollary of the following theorem.

Theorem 0 Let Σ be an interval in $\mathbb{R}$ (not necessarily bounded) containing 0 in its interior. Suppose $\pi : \Sigma \times \Sigma \to [0, 1]$ satisfies the following conditions for any $\lambda \in [0, 1]$ and any $x, y, x', y', z \in \Sigma$:

(i) $\pi(x, y) = \frac{1}{2}$ iff $x = y$

(ii) $\pi(x, y) + \pi(y, x) = 1$

(iii) $\pi(x, y) = \pi(x', y')$ implies

$$\pi(x\lambda z, y\lambda z) = \pi(x'\lambda z, y'\lambda z)$$

(iv) π is non-decreasing (respectively, non-increasing) in its first (respectively, second) argument

(v) π is continuous in each argument

Then

$$\pi(x, y) \geq \pi(x', y') \quad \text{iff} \quad x - y \geq x' - y' \tag{5}$$

for any $x, y, x', y' \in \Sigma$.

The proof of Theorem 0 may be found in Appendix A.

To interpret Theorem 0, suppose we have a representation $u : A \to \mathbb{R}$ for the preference relation $\succsim^P$ defined by (2). Let $\Sigma = u(A)$. Assuming $0 \in \text{int}(\Sigma)$ is without loss of generality (WLOG) provided preferences are non-trivial (Σ is not a singleton). Now re-calibrate P in terms of utility pairs by defining a mapping $\pi : \Sigma \times \Sigma \to [0,1]$ as follows: $\pi(x,y) = P(a,b)$ for any $a,b \in A$ with $u(a) = x$ and $u(b) = y$. *Provided this mapping is well-defined and satisfies (i)–(v)*, Theorem 0 implies that (3) holds. Corollary 0 establishes sufficient conditions for the italicised provisos to be met. In particular, Strong M-Independence ensures (iii).[16]

Corollary 0 Let A be a mixture set and $M \subseteq A$, and let P satisfy SST, D2 and Strong M-Independence. If $u : A \to \mathbb{R}$ is M-linear and represents $\succsim^P$ defined by (2), then (3) holds for any $a, b, c, d \in A$.

Corollary 0 is the main result of the paper. It is proved in Appendix A.

Corollary 0 suggests the following "recipe" for producing strong M-linear utility representation theorems.

Step I. Choose a set of axiomatic restrictions on $\succsim^P$ that are sufficient for an M-linear utility representation. Translate these axioms into the corresponding restrictions on P using (2).

Step II. Add SST, D2 and Strong M-Independence, then apply Corollary 0.

In most cases, the axioms required at Step II already imply certain of the restrictions needed for Step I, allowing for some economies to be achieved.

Of course, for this recipe to be useful from a normative point of view, Strong M-Independence must be a reasonable restriction on P. If we assume – as seems natural – that the binary relation $\succsim^P$ defined by (2) orders the (utility) stimuli experienced by the agent, then we may argue as follows.

[16] Note that the function π induced by Example 1 violates property (iii). Suppose, for example, that $\overline{u} = 1$ and $\underline{u} = 0$. Let $x = 1$, $x' = \frac{1}{2}$ and $y = z = 0$, and note that

$$\pi(x,0) = \pi(x',0) = \frac{3}{4}$$

while

$$\pi\left(\frac{x}{2},0\right) = \frac{3}{4} \neq \frac{5}{8} = \pi\left(\frac{x'}{2},0\right).$$

The axioms imposed at Step I embody an implicit argument that $\succsim^P$ should possess an M-linear representation. If so, then $\succsim^P$ satisfies the following condition (which might be called M-Independence): for any $a, b \in A$, any $c \in M$ and any $\lambda \in (0, 1)$,

$$a \succsim^P b \quad \Rightarrow \quad a\lambda c \succsim^P b\lambda c.$$

In other words, if option a generates a stimulus at least as great as option b, then this remains true after mixing with a common element from M. For example, it may be that elements from M do not (or should not) "interfere" with other stimuli under mixing, so options are (or should be) compared by first excluding common elements mixed in from M. By this logic – which need not be compelling in all circumstances – we have reason to expect that choice probabilities will (or should) be unaffected by mixing with a common element from M.

4 Applications

The following sections apply our recipe to prove strong M-linear utility representation theorems for a variety of specifications of A, M and the mixture operation $a\lambda b$. These provide selective illustrations only – other applications are no doubt possible and potentially of interest.

We first consider the case $M = A$, so u is (mixture-)linear *simpliciter.* When A is the unit simplex in $\mathbb{R}^n$ and the usual mixture operation is assumed, we are in the domain of Dagsvik (2008). Our recipe therefore provides an alternative axiomatisation of the strong expected utility representation, and an alternative proof of the representation theorem. If A is a set of *acts* of the Anscombe-Aumann (AA) variety and mixtures are defined as in Anscombe and Aumann (1963), we obtain a stochastic extension of AA version of subjective expected utility (SEU) theory.

We next consider three examples in which M is a proper subset of A, obtaining stochastic extensions of Dual Theory (DT), Choquet expected utility (CEU) theory and maxmin expected utility (MEU) theory.

In proving these results, most of the effort is expended in eliminating redundancies in the axioms introduced at Steps I and II. More direct applications of the recipe would yield shorter proofs but longer lists of axioms in the statements of propositions.

For future reference, we define the following natural strengthening of Axiom D2, which will be useful in the sequel:

Mixture Solvability (MS) For all $a, b, c \in A$ and all $\pi \in (0, 1)$

$$P(a, b) \geq \pi \geq P(a, c) \quad \Rightarrow \quad P(a, b\lambda c) = \pi \text{ for some } \lambda \in [0, 1]$$

The MS condition, together with Strong M-Independence, ensures that the binary relation (2) possesses sufficient "continuity" to make Step I in the examples below.[17]

Note that if A in Example 1 is a mixture set and $\hat{u}$ is (mixture-)linear, then MS is satisfied. It follows that strengthening D2 to MS does not, on its own, allow us to weaken the quadruple condition to SST in Debreu's theorem. This observation is significant, as our representation theorems will use SST rather than the stronger quadruple condition.

4.1 Strong expected utility

Let us first recall Dagsvik's (2008) representation theorem.

In addition to Axioms D1, D2 and Strong Independence, Dagsvik also requires:

Axiom D3 [**Archimedean Property**] For all $a, b, c \in A$, if

$$P(a, b) > \frac{1}{2} > P(c, b)$$

then there exist $\alpha, \beta \in (0, 1)$ such that

$$P(a\alpha c, b) > \frac{1}{2} > P(a\beta c, b).$$

Theorem D [**Dagsvik, 2008, Theorems 2 and 4**] If A is the unit simplex in $\mathbb{R}^n$ and P satisfies D1-D3 and Strong Independence, then P has a strong expected utility representation.

Corollary 0 may be applied to give a variation on Theorem D; one which is applicable to any mixture set. It uses SST in place of the stronger quadruple condition (Axiom D1). Note that several economies are possible at Step I. The restrictions on P imposed by transitivity and mixture-independence of the binary relation $\succsim^P$ are guaranteed by SST and Stong Independence respectively. We invoke MS and Strong Independence to ensure that P also satisfies the restrictions implied by a standard continuity condition on $\succsim^P$ (Lemma 1).

[17] See Lemma 1.

Proposition 1 If A is a mixture set and P satisfies SST, MS and Strong Independence, then P has a strong expected utility representation (i.e., representation (3) for some linear u).

Proof. Let $\succsim^P$ be the binary relation defined by (2). For Step I it suffices to prove that $\succsim^P$ has an A-linear (i.e., linear) representation.

It is obvious that $\succsim^P$ is complete. It is transitive by SST. Setting $a' = b'$ in the definition of Strong Independence, we see that $\succsim^P$ inherits the following independence property: for all $a, b, c \in A$ and all $\lambda \in (0, 1)$

$$a \succsim^P b \quad \Rightarrow \quad a\lambda c \succsim^P b\lambda c \tag{6}$$

Finally, $\succsim^P$ satisfies a standard continuity property.

Lemma 1 For any $a, b, c \in A$, the sets

$$\left\{\lambda \in [0,1] \;\middle|\; a\lambda b \succsim^P c\right\}$$

and

$$\left\{\lambda \in [0,1] \;\middle|\; c \succsim^P a\lambda b\right\}$$

are closed.

Proof. The basic idea of the proof is simple. First, we use (6) to show that preferences are monotone in λ. Second, MS implies that preferences exhibit solvability with respect to λ. Finally, we observe that monotonicity plus solvability implies continuity in λ.

Fix some $a, b, c \in A$ and consider the set

$$\left\{\lambda \in [0,1] \;\middle|\; c \succsim^P a\lambda b\right\} \tag{7}$$

It is without loss of generality (WLOG) to assume $a \succsim^P b$. Then $a\lambda b \succsim^P b$ for any $\lambda \in (0, 1)$ by (6). Hence, applying (6) once more, we have:

$$a\lambda b \succsim^P b\mu\,(a\lambda b)$$

for any $\mu \in (0, 1)$. Since $b\mu\,(a\lambda b) = a\,[\lambda\,(1-\mu)]\,b$, it follows that if λ is in (7) then so is any $\lambda' < \lambda$. It remains to exclude the possibility that (7) is of the form $[0, \zeta)$ for some $\zeta > 0$.

Let $\{\lambda_m\}_{m=1}^{\infty} \subseteq [0, \zeta)$ be a convergent sequence with limit ζ. Suppose $a\lambda_m b \succsim^P c$ for each m, while $c \succ^P a\zeta b$. Then

$$P\,(c, a\zeta b) \;>\; \frac{1}{2} \;\geq\; P\,(c, a\lambda_m b)$$

for each m. Let

$$x \in \left(P\left(c, a\zeta b\right), \frac{1}{2} \right).$$

Given some $m \in \{1, 2, ...\}$, MS ensures that there exists $\mu \in (0,1)$ such that

$$x \;=\; P\left(c, \left(a\zeta b\right) \mu \left(a\lambda_m b\right)\right) \;=\; P\left(c, a\left[\mu\zeta + \left(1-\mu\right)\lambda_m\right] b\right).$$

Since $\mu\zeta + (1-\mu)\lambda_m \in [0,\zeta)$ and $x > \frac{1}{2}$ we have the desired contradiction.

The same sort of argument (*mutatis mutandis*) shows that the set

$$\left\{ \lambda \in [0,1] \;\middle|\; a\lambda b \succsim^P c \right\}$$

is closed. □

By Theorem 1 of Fishburn (1982, Chapter 2), $\succsim^P$ possesses a linear representation, u. The result now follows by Corollary 0. In particular, Strong Independence is equivalent to Strong A-Independence. □

Proposition 1 provides a more "compact" axiomatisation of strong expected utility than Theorem D, in the sense of having one fewer axioms (D3 is not required) and using SST in place of the stronger QC. However, it also uses a strengthened version of solvability (MS rather than D2), so the relationship between the two sets of axioms is not transparent. Appendix B offers some clarification. It proves a strengthened version of Theorem D, called Theorem D*, in which SST replaces Axiom D1 (QC). Proposition 1 and Theorem D* together show that MS is equivalent to the conjunction of D2 and D3, given SST and Strong Independence.

The proof of Theorem D* follows our usual recipe, but a more elaborate argument is required at Stage I, since we cannot use MS to establish the required continuity of $\succsim^P$ (Lemma 1). Instead, the argument leans more heavily on Strong Independence, which, it turns out, imposes restrictions on $\succsim^P$ over and above the independence condition (6).

4.2 Preferences over AA acts

Consider an Anscombe-Aumann environment in which A is a set of acts (uncertain prospects) with the following structure: there is a finite set S of states and a mixture set C of consequences, and A consists of all functions from S to C. The mixture operation on C induces a mixture operation on A in the usual (Anscombe-Aumann) manner: given $a, b \in A$ and $\lambda \in [0,1]$, the function $a\lambda b \in A$ maps state $s \in S$ to consequence $a(s)\,\lambda b(s) \in C$.

We use A^c to denote the set of constant functions in A, and take the usual notational liberty of identifying A^c with C. We also define

$$\Delta(S) = \left\{\theta : S \to [0,1] \;\middle|\; \sum_s \theta(s) = 1\right\}$$

to be the set of all possible probability distributions over the states.

Anscombe and Aumann (1963) take C to be the unit simplex in $\mathbb{R}^n$, interpreted as the set of roulette lotteries over a fixed set of n prizes and endowed with the usual mixing operation, but this is not essential to our analysis.[18]

4.2.1 Subjective expected utility

A utility function $u : A \to \mathbb{R}$ has the SEU structure if it takes the form

$$u(a) = \sum_{s \in S} \theta(s)\, v(a(s)) \tag{8}$$

for some linear function $v : C \to \mathbb{R}$ and some $\theta \in \Delta(S)$. Note that u in (8) is A-linear, but not every A-linear function has the form (8). If (3) holds with u of the SEU form, we say that P has a *strong SEU representation.*

We may apply our recipe to obtain a strong SEU representation theorem. Several economies are possible in specifying the axioms. Given the axioms introduced at Step II, we need only add the following in order to make Step I:[19]

Stochastic State Monotonicity For any $a, b \in A$, if

$$P(a(s), b(s)) \geq \frac{1}{2}$$

for every $s \in S$ then $P(a,b) \geq \frac{1}{2}$.

Proposition 2 If P satisfies SST, MS, Stochastic State Monotonicity and Strong Independence, then P has a strong SEU representation.

Proof. By Corollary 0, it suffices to show that $\succsim^P$ has a SEU representation.

[18] That is, the formalities rely only on C being a mixture set. At the level of interpretation matters are not so robust, since axioms such as Strong Independence depend for their plausibility on C being a set of roulette lotteries.

[19] Recall that we identify the *consequence* $p \in C$ with the *constant function* in A^c that maps each state to p.

Observe that A^c is a mixture set and the restriction of P to $A^c \times A^c$ satisfies SST, Strong Independence and MS. From the proof of Proposition 1, we therefore deduce that the restriction of $\succsim^P$ to A^c is a weak order satisfying (6) and (7), and hence possesses a linear representation. Denote the latter by $v : C \to \mathbb{R}$ and let $\Lambda = v(C)$. Then Λ is a closed interval. Given Stochastic State Monotonicity, the result is trivial if Λ is a singleton, so assume otherwise.

Consider the following binary relation $\geq^*$ on the set Λ^S of state-utility vectors: $x \geq^* y$ if $a \succsim^P b$ for some $a, b \in A$ with $x_s = v(a(s))$ and $y_s = v(b(s))$ for each $s \in S$. This binary relation is well-defined by Stochastic State Monotonicity: for any $a, b \in A$

$$a(s) \succsim^P b(s) \text{ for all } s \in S \quad \Rightarrow \quad a \succsim^P b \tag{9}$$

Let $>^*$ denote the asymmetric part of $\geq^*$.

It is straightforward to verify (using SST and Strong Independence) that $\geq^*$ is a weak order that satisfies

$$x \geq^* y \quad \Rightarrow \quad \lambda x + (1-\lambda) z \geq^* \lambda y + (1-\lambda) z$$

for all $x, y, z \in \Lambda^S$ and all $\lambda \in (0,1)$. To prove that $\geq^*$ has a linear representation, it suffices (see the proof of Proposition 1) to show:

Lemma 2 The following sets are closed for all $x, y, z \in \Lambda^S$:

$$\{\mu \in [0,1] \mid \mu x + (1-\mu) y \geq^* z\}$$

and

$$\{\mu \in [0,1] \mid z \geq^* \mu x + (1-\mu) y\}.$$

Proof. Define $P^* : \Lambda^S \times \Lambda^S \to [0,1]$ as follows: $P^*(x,y) = P(a,b)$ if there exists $a, b \in A$ with $x_s = v(a(s))$ and $y_s = v(b(s))$ for each $s \in S$. This is well-defined, since weak substitutability implies $P(a,b) = P(a',b')$ whenever $a \sim^P a'$ and $b \sim^P b'$, and the state monotonicity property (9) means that $a \sim^P a'$ whenever $a(s) \sim^P a'(s)$ for all $s \in S$. Thus, $x \geq^* y$ iff $P^*(x,y) \geq \frac{1}{2}$.

Note that P^* inherits the MS property from P: for any $x, y, z \in \Lambda$, any $\pi \in (0,1)$ and any $a, b, c \in A$ with $x_s = v(a(s))$, $y_s = v(b(s))$ and $z_s = v(c(s))$ for each $s \in S$:

$$\begin{aligned} P^*(x,y) \geq \pi \geq P^*(x,z) \quad &\Rightarrow \quad P(a,b) \geq \pi \geq P(a,c) \\ &\Rightarrow \quad P(a, b\lambda c) = \pi \quad \text{for some } \lambda \in [0,1] \\ &\Rightarrow \quad P^*(x, \lambda y + (1-\lambda) z) = \pi \quad \text{for some } \lambda \in [0,1] \end{aligned}$$

where the final inequality uses the linearity of v.

Lemma 2 now follows by the same argument as we used to prove Lemma 1. □

Thus, $\geq^*$ has a linear representation: there exists $\theta : S \to \mathbb{R}$ such that

$$x \geq^* y \quad \Leftrightarrow \quad \sum_s \theta(s) x_s \geq \sum_s \theta(s) y_s$$

for any $x, y \in \Lambda^S$. Using (9) and the fact that Λ is a non-degenerate interval, it is easily verified that $\theta(s) \geq 0$ for all s and strictly so for at least one. We may therefore normalise θ so that it lies in $\Delta(S)$. □

4.2.2 Choquet expected utility

An important generalisation of SEU is the *Choquet expected utility (CEU)* model of Schmeidler (1989). If there exists a utility function $u : A \to \mathbb{R}$ of the CEU form (defined below) such that (3) holds for any $a, b, c, d \in A$, we say that P has a *strong CEU representation*.

To discuss CEU we need a couple of additional definitions.

Mappings $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are *comonotonic* if there do *not* exist states $s, s' \in S$ with $f(s) > g(s)$ and $g(s') > f(s')$. Analogously, given a weak order $\succsim$ on A we say that acts $a, b \in A$ are $\succsim$-comonotonic if there do not exist states $s, s' \in S$ with $a(s) \succ b(s)$ and $b(s') \succ a(s')$, where $\succ$ is the asymmetric part of $\succsim$. The notion of comonotonicity is central to the CEU model.

A *capacity* on S is a mapping $\omega : 2^S \to [0,1]$ that satisfies $\omega(\emptyset) = 0$, $\omega(S) = 1$ and $\omega(A) \leq \omega(B)$ whenever $A \subseteq B$. Capacities are non-additive generalisations of probability measures.

A utility function $u : A \to \mathbb{R}$ has the CEU form if there exists a linear function $v : C \to \mathbb{R}$ and a capacity $\omega : 2^S \to [0,1]$ such that

$$u(a) = \int_{-\infty}^{0} [\omega(\{s \in S \mid v(a(s)) > z\}) - 1] \; dz + \int_{0}^{\infty} \omega(\{s \in S \mid v(a(s)) > z\}) \; dz$$

We may write this more compactly using the Choquet integral (Denneberg, 1994):

$$u(a) = \int (v \circ a) \; d\omega \tag{10}$$

If ω is additive – a probability measure – then (10) is just the usual expected value of $v \circ a$ with respect to ω. It is well known (Denneberg, 1994) that the Choquet integral is *homogeneous* and *comonotonically additive*. That is, given any two comonotonic functions f and g, any $\alpha > 0$ and any capacity ω:

$$\int \alpha f \; d\omega \;=\; \alpha \int f \; d\omega.$$

and

$$\int (f+g) \; d\omega \;=\; \int f \; d\omega \;+\; \int g \; d\omega.$$

Suppose the binary choice probability function P is such that $\succsim^P$ has CEU representation (10). Then acts $a, b \in A$ are $\succsim^P$-comonotonic iff the mappings $v \circ a$ and $v \circ b$ are comonotonic. The linearity of v therefore implies that

$$u(a\lambda b) \;=\; \lambda u(a) + (1-\lambda) u(b)$$

whenever $a, b \in A$ are $\succsim^P$-comonotonic.

Since $b \in A^c$ is $\succsim^P$-comonotonic with any $a \in A$, the CEU utility function (10) is A^c-linear.[20] Strong A^c-Independence is therefore necessary for the existence of a strong CEU representation. However, Strong A^c-Independence, together with SST, MS and Stochastic State Monotonicity, do not suffice.[21] We need the following additional condition to complete Step I:[22]

Stochastic Comonotonic Independence For any pairwise $\succsim^P$-comonotonic acts $a, b, c \in A$ and any $\lambda \in (0,1)$,

$$P(a,b) \geq \frac{1}{2} \quad \Rightarrow \quad P(a\lambda c, b\lambda c) \geq \frac{1}{2}.$$

Proposition 3 If P satisfies SST, MS, Strong A^c-Independence, Stochastic State Monotonicity and Stochastic Comonotonic Independence, then P has a strong CEU representation.

[20] One may verify that $u(A^c) = u(A)$ as follows. The function

$$F(z) = \omega(\{s \in S \mid v(a(s)) \leq z\})$$

is the cumulative distribution function for a random variable whose range is contained in $v(C)$ and (10) is the expected value of this random variable. It follows that $u(A) \subseteq v(C) = u(A^c)$.

[21] Consider preferences $\succsim \subseteq A \times A$ which have a maxmin expected utility representation $u : A \to \mathbb{R}$ (see Section 4.2.3) but no CEU representation. Let $Z = u(A) - u(A)$ and let $P(a,b) = g(u(a) - u(b))$ for some continuous and strictly increasing function $g : Z \to [0,1]$ with $g(z) + g(-z) = 1$. Note that $\succsim = \succsim^P$. Then P satisfies SST, MS, Strong A^c-Independence and Stochastic State Monotonicity, but it does not have a strong CEU representation: if it did, then $\succsim^P$ would have a CEU representation.

[22] Stochastic Comonotonic Independence is the restriction on P imposed by *comonotonic independence* (Schmeidler, 1989) of the binary relation $\succsim^P$.

Proof. We first show that $\succsim^P$ defined by (2) has a CEU representation. Our argument follows the structure in Ryan (2009; Section 5).

As in the proof of Proposition 2, we first consider the restriction of P to $A^c \times A^c$ and the associated restriction of $\succsim^P$ to A^c. By the same argument as in the earlier proof,[23] $\succsim^P$ restricted to A^c has a linear representation, which we denote $v : C \to \mathbb{R}$. As before, we let $\Lambda = v(C)$, a closed interval. The result is trivial if Λ is a singleton, so we assume otherwise henceforth.

Define the binary relation $\geq^*$ on Λ^S as before: $x \geq^* y$ if $a \succsim^P b$ for some $a, b \in A$ with $x_s = v(a(s))$ and $y_s = v(b(s))$ for each $s \in S$.

For each permutation $\rho : S \to \{1, 2, ..., |S|\}$ we define the associated (convex) *comonotonicity region* $\mathcal{C}(\rho)$ as in Ryan (2009): $x \in \mathcal{C}(\rho)$ iff

$$x_{\rho(1)} \geq x_{\rho(2)} \geq \cdots \geq x_{\rho(|S|)}.$$

Let $\geq^*_\rho$ denote $\geq^*$ restricted to $\mathcal{C}(\rho)$. Strong Comonotonic Independence ensures that each $\geq^*_\rho$ satisfies

$$x \geq^*_\rho y \quad \Rightarrow \quad \lambda x + (1-\lambda) z \geq^*_\rho \lambda y + (1-\lambda) z$$

for all $x, y, z \in \mathcal{C}(\rho)$ and all $\lambda \in [0,1]$. We may therefore follow the same logic as in the proof of Proposition 2 to show that each $\geq^*_\rho$ has a linear representation. That is, for each permutation ρ, there exists $\theta^\rho \in \Delta(S)$ such that

$$x \geq^*_\rho y \quad \Leftrightarrow \quad \sum_s \theta^\rho(s)\, x_s \;\geq\; \sum_s \theta^\rho(s)\, y_s \tag{11}$$

for any $x, y \in \mathcal{C}(\rho)$. The argument in Ryan (2009, pp.345-7) now shows that there exists a capacity $\omega : 2^S \to [0,1]$ such that $x \geq^* y$ iff

$$\int_{-\infty}^{0} \left[\omega(\{s \in S \mid x_s > z\}) - 1\right] \, dz \; + \; \int_{0}^{\infty} \omega(\{s \in S \mid x_s > z\}) \, dz \; \geq$$

$$\int_{-\infty}^{0} \left[\omega(\{s \in S \mid y_s > z\}) - 1\right] \; + \; \int_{0}^{\infty} \omega(\{s \in S \mid y_s > z\}) \, dz$$

This gives the desired CEU representation for $\succsim^P$.

To complete the proof, apply Corollary 0. □

[23] In particular, Strong A^c-Independence of P ensures that the restriction of P to $A^c \times A^c$ satisfies Strong Independence.

4.2.3 Maxmin expected utility

A utility function $u : A \to \mathbb{R}$ has the *maxmin expected utility (MEU)* form if there exists a linear function $v : C \to \mathbb{R}$ and a closed and convex set $\mathcal{P} \subseteq \Delta(S)$ such that

$$u(a) \;=\; \min_{\theta \in \mathcal{P}} \sum_{s \in S} \theta(s)\, v(a(s)) \tag{12}$$

(Gilboa and Schmeidler, 1989). Note that (12) is A^c-linear.

To obtain a SUR with u of the MEU form, we shall require:

Stochastic Uncertainty Aversion For any $a, b \in A$ and any $\lambda \in (0, 1)$,

$$P(a, b) = \frac{1}{2} \quad \Rightarrow \quad P(a\lambda b, b) \geq \frac{1}{2}.$$

Proposition 4 If P satisfies SST, MS, Stochastic State Monotonicity, Stochastic Uncertainty Aversion and Strong A^c-Independence, then P has a strong utility representation with respect to some u of the MEU form (12).

Proof. As for the proof of Proposition 2, we may show that the restriction of $\succsim^P$ to A^c possesses a linear representation, $v : C \to \mathbb{R}$. We let $\Lambda = v(C)$ and exclude the trivial cases in which Λ is a singleton. We may therefore assume (WLOG) that Λ contains 0 in its interior. The binary relation $\geq^*$ on Λ^S is defined in the usual way, with asymmetric part $>^*$ and symmetric part $=^*$.

We also identify Λ with the set of constant vectors in Λ^S.

Note that $\geq^*$ is a weak order that satisfies the following conditions for any $x, y \in \Lambda^S$, any $k \in \Lambda$ and any $\lambda \in (0, 1)$:

$$x \geq^* y \quad \Rightarrow \quad \lambda x + (1 - \lambda) k \geq^* \lambda y + (1 - \lambda) k \tag{13}$$

$$x =^* y \quad \Rightarrow \quad \lambda x + (1 - \lambda) y \geq^* x \tag{14}$$

Furthermore, Stochastic State Monotonicity implies that $\geq^*$ satisfies the following *strict monotonicity* condition: for any $x, y \in \Lambda^S$ and any $k, k' \in \Lambda$,

$$x \geq y \quad \Rightarrow \quad x \geq^* y \tag{15}$$

and

$$k > k' \quad \Rightarrow \quad k >^* k' \tag{16}$$

We next prove that there exists a closed and convex set $\mathcal{P} \subseteq \Delta(S)$ such that $\geq^*$ is represented by

$$u(x) = \min_{\theta \in \mathcal{P}} \sum_{s \in S} \theta(s) x_s \tag{17}$$

The result will then follow by Corollary 0. The following lemmata establish the required representation (17).

Lemma 3 For any $x \in \Lambda^S$, there exists a unique $k \in \Lambda$ such that $x =^* k$.

Proof. Define P^* as in the proof of Lemma 2, and note that it satisfies mixture solvability by the same argument as before. Let $\underline{x} = \min_{s \in S} x_s$ and $\overline{x} = \max_{s \in S} x_s$. Then (15) implies

$$P^*(x, \underline{x}) \geq \frac{1}{2} \geq P^*(x, \overline{x}).$$

Mixture solvability ensures that $x =^* \lambda \underline{x} + (1-\lambda)\overline{x}$ for some $\lambda \in [0,1]$. Uniqueness follows by (16). □

In view of Lemma 3, for each $x \in \Lambda$ we let k^x denote the unique $k \in \Lambda$ satisfying $x =^* k$.

Lemma 4 For any $x, y \in \Lambda^S$, any $k \in \Lambda$ and any $\lambda \in (0,1)$:

$$x >^* y \quad \Rightarrow \quad \lambda x + (1-\lambda) k >^* \lambda y + (1-\lambda) k$$

Proof. If $x >^* y$ then $\lambda k^x + (1-\lambda) k > \lambda k^y + (1-\lambda) k$. Since

$$\lambda x + (1-\lambda) k =^* \lambda k^x + (1-\lambda) k$$

and

$$\lambda y + (1-\lambda) k =^* \lambda k^y + (1-\lambda) k$$

by (13), the result follows. □

Combining this result with (13) we have:

Corollary 1 For any $x, y \in \Lambda^S$, any $k \in \Lambda$ and any $\lambda \in (0,1)$:

$$x \geq^* y \quad \Leftrightarrow \quad \lambda x + (1-\lambda) k \geq^* \lambda y + (1-\lambda) k \tag{18}$$

Lemma 5 There is a closed, convex cone $K \subseteq \mathbb{R}^S$ such that

$$\left\{x \in \Lambda^S \;\middle|\; x \geq^* 0\right\} \;=\; K \cap \Lambda^S.$$

Moreover, for any $k \in \Lambda$,

$$\left\{x \in \Lambda^S \;\middle|\; x \geq^* k\right\} \;=\; (K + \{k\}) \cap \Lambda^S.$$

Proof. Lemma 2 and Corollary 2 in Ryan (2009) ensure the existence of a suitable convex cone K. It remains to show that K is closed, for which it suffices to show that the set

$$\left\{x \in \Lambda^S \;\middle|\; x \geq^* 0\right\} \tag{19}$$

is closed, since 0 lies in the interior of Λ.

Suppose there exists a sequence $\{x^m\}_{m=1}^{\infty} \subseteq \Lambda^S$ with limit $\hat{x} \in \Lambda^S$ such that $x^m \geq^* 0$ for all m and $0 >^* \hat{x}$. Since the set (19) is convex, $\hat{x}$ must be a boundary point. Let $\hat{k} > 0$ be from the interior of Λ. Then $\hat{k} \in \Lambda^S$ is in the interior of (19) and we have

$$P^*(0, \hat{x}) \;>\; \frac{1}{2} \;\geq\; P^*\left(0, \hat{k}\right).$$

Given $\pi \in \left(\frac{1}{2}, P\left(0, \hat{k}\right)\right)$ there exists some $\lambda \in (0, 1)$ such that

$$P^*\left(0, \hat{x}\lambda\hat{k}\right) \;=\; \pi.$$

But $\hat{x}\lambda\hat{k}$ must lie in the set (19), which is a contradiction. □

From Lemma 5 and the fact that $\mathbb{R}^S_+ \subseteq K$ the desired representation (17) follows by the argument on p.338 of Ryan (2009).

This completes the proof of Proposition 4. □

4.3 Dual theory

For our final application we return to the domain of risk. We consider a SUR whose core theory is Yaari's (1987) special case of the rank-dependent expected utility (RDEU) model, also known as the Dual Theory (DT). To expedite the analysis, we exploit the close relationship between CEU and RDEU, originally pointed out by Wakker (1990).

Let us therefore adopt the set-up of Section 4.2.2, but with consequences specified as amounts of money, so that $C = \mathbb{R}_+$ (endowed with the usual mixture operation). We also endow S with a probability measure p so that acts embody risk rather than uncertainty. Thus, elements of A are real-valued random variables. We let F^a denote the distribution function of the random variable $a \in A$.

Because of the finiteness of S, this structure induces a rather limited range of distributions on C. It is therefore conventional to work with a richer set of states (such as $S = [0,1]$) when studying risk. We could have done so here. However, the more restrictive environment allows us to leverage off our efforts in Section 4.2.2. The reader who is interested in the elaborations necessary to accommodate a richer state space is referred to Ryan (2009, Appendix 5 of the Supplementary Material).

Given our current specification of A, a utility function $u : A \to \mathbb{R}$ has the RDEU form if it satisfies (10) with $\omega = \phi \circ p$ for some non-decreasing and surjective function $\phi : [0,1] \to [0,1]$.[24] It has the DT form if, in addition, v is linear. Since the A^c-linearity of (10) relies on the linearity of $v : C \to \mathbb{R}$, we can only use our recipe to establish a strong utility representation with DT at its core, rather than the more general RDEU theory.

We will require the following strengthening of Stochastic State Monotonicity.

Stochastic Monotonicity For any $a, b \in A$, if F^a first-order stochastically dominates F^b, then

$$P(a, b) \geq \frac{1}{2}.$$

Proposition 5 If P satisfies SST, MS, Stochastic Monotonicity, Strong A^c-Independence and Stochastic Comonotonic Independence, then P has a strong utility representation with respect to some u of the DT form.

Proof. We first show that $\succsim^P$ defined by (2) has a DT representation. This binary relation shares the same properties as $\geq^*$ in the proof of Proposition 3. By the argument in that proof, $\succsim^P$ therefore possesses a CEU representation

$$u(a) = \int a \, d\omega$$

for some capacity ω. By Stochastic Monotonicity, $\omega = \phi \circ p$ for some non-decreasing and surjective mapping $\phi : [0,1] \to [0,1]$ (Wakker, 1990).

Since u is A^c-linear, we may apply Corollary 0 to complete the proof. $\square$

[24] It is easily checked that $\phi \circ p$ is a capacity.

5 Concluding remarks

Much of the empirical literature on binary stochastic choice uses experimental data in which alternatives embody risk or uncertainty. Non-(S)EU forms for utility are often tested against (S)EU comparators. The present paper provides sufficient conditions for the validity of several such models.

Despite their simple – and hence restrictive – form, Fechner models continue to be workhorses for the empirical analysis of choice behaviour. Amongst the Fechner models, the strong utility form (3) is almost always employed, albeit implicitly. To see why, note that empirical implementations of Fechner models typically assume

$$P(a,b) \;=\; \Pr\left[X \leq u(a) - u(b)\right]$$

for some random variable X that is symmetrically distributed about zero, with a distribution function that is continuous and *strictly increasing* (such as the Normal distribution function). That is:

$$P(a,b) \;=\; F\left[u(a) - u(b)\right] \tag{20}$$

for some continuous and strictly increasing distribution function $F : Z \to [0,1]$ satisfying $\phi(z) + \phi(-z) = 1$, where Z is the interval (symmetric about zero) on which the distribution is supported. Since F is strictly increasing, specification (20) implies a strong utility representation.[25]

It is therefore important to understand the axiomatic foundations of strong utility models for choices between risky or uncertain prospects. It is also important to understand the axiomatic distinctions between strong utility models with different core theories. In an early (and elegant) empirical test, Loomes and Sugden (1998) rejected a strong expected utility representation (SEUR) for their experimental data. They argue that the fault lies predominantly with the assumption of linear utility rather than the strong utility model. In a similar vein, Loomes and Pogrebna (2014) reject the stochastic version of the *independence* axiom (Dagsvik, 2008) in their data. Recalling Hey's (1995) quote from the Introduction, embedding EU in a strong utility structure does not make it empirically respectable in an absolute sense – the question is one of relative performance against alternative core theories.

[25] Continuity of F further implies that mixture solvability (MS) is satisfied. For any of the core theories in Propositions 1-5, our axioms are therefore *necessary* as well as sufficient for the Fechner models typically assumed in empirical work.

Of course, our axiomatisations are restricted to M-linear utility classes and therefore exclude important core theories, such as rank-dependent expected utility (RDEU).[26] Filling these gaps is an important item for the future research agenda.

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Appendices

A Proofs of Theorem 0 and Corollary 0

A.1 Proof of Theorem 0

To prove Theorem 0, suppose that $\pi : \Sigma \times \Sigma \to [0,1]$ satisfies (i)–(v). Let us define

$$\Lambda = \{(x,y) \in \Sigma \times \Sigma \mid x - y = 0\}$$

for ease of reference. Property (i) says that Λ is the level curve corresponding to $\pi = \frac{1}{2}$. We need to show that all the other level curves are parallel to this one.

First observe that we may strengthen the monotonicity condition (iv).[27]

Lemma A The function π is *strictly* increasing (respectively, decreasing) in its first (respectively, second) argument.

Proof. Suppose $x > x'$ but $\pi(x, y) = \pi(x', y)$. Properties (i) and (iv) imply that $x' > y$ or $y > x$. Assume the former (i.e., $y > x > x'$), the latter case being handled similarly. Thus

$$\pi(x, y) = \pi(x', y) > \frac{1}{2} \tag{21}$$

from (i) and (iv).

Define $x^0 = x$, $x^1 = x'$ and

$$\lambda = \frac{x^1 - y}{x^0 - y},$$

so $x^1 = x^0 \lambda y$. For each $k \in \{2, 3, ...\}$ let

$$x^k = x^{k-1} \lambda y = x^0 \lambda^k y.$$

Since $\lambda \in (0, 1)$ we have $x^k \to y$ as $k \to \infty$. By (v) and (i), we therefore have

$$\lim_{k \to \infty} \pi\left(x^k, y\right) = \pi(y, y) = \frac{1}{2} \tag{22}$$

Since $\pi(x^0, y) = \pi(x^1, y)$, successive applications of (iii) give

$$\pi\left(x^k, y\right) = \pi(x, y)$$

for all k, so (21) and (22) deliver the required contradiction. $\square$

Now suppose, contrary to what we seek to show, that there exist $x, y, x', y' \in \Sigma$ with

$$x - y = x' - y' \neq 0$$

but

$$\pi(x, y) > \pi(x', y').$$

It suffices to consider the case[28]

$$x - y = x' - y' < 0.$$

[27] Since $\pi(x, y) \geq \frac{1}{2}$ iff $x \geq y$ from (i), Lemma A implies that π exhibits the following *substitutability* property:

$$\pi(x, y) \geq \frac{1}{2} \quad \text{iff} \quad \pi(x, z) \geq \pi(y, z).$$

[28] The other case follows directly, since $\pi(x, y) > \pi(x', y')$ implies $\pi(y, x) < \pi(y', x')$ by virtue of (ii).

Thus, (x, y) and (x', y') are distinct points lying above Λ and on a line parallel to it. See Figure 1 (which assumes $x' > x$).

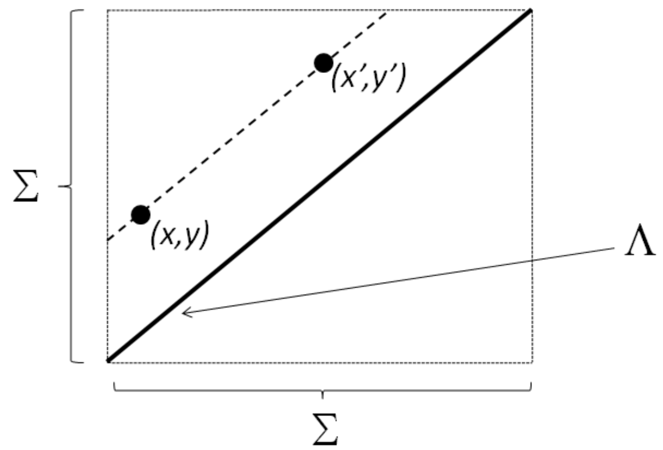

Figure 1

From (i) and Lemma A we have

$$\pi(x', y') < \pi(x, y) < \pi(y, y) = \frac{1}{2} = \pi(x', x').$$

Therefore, using (v) and Lemma A, we have $\pi(x, y) = \pi(x', y'')$ for some $y'' \in (x', y')$. Since the line joining (x, y) and (x', y'') is not parallel to Λ we have

$$(x', y'') = (x\lambda z, y\lambda z) \tag{23}$$

for some $z \in \mathbb{R}$ and some $\lambda \in (0, 1)$. Figure 2 illustrates (again for the case $x < x'$). We may assume that $z \in \Sigma$; otherwise, we can use (iii) to contract (x, y) and (x', y'') towards $(0, 0)$ until this condition is satisfied.[29]

Starting from (23), repeated applications of (iii) give that

$$\frac{1}{2} > \pi(x, y) = \pi(x\lambda^n z, y\lambda^n z) \qquad \text{for each } n \in \{1, 2, ...\} \tag{24}$$

Since $(x\lambda^n z, y\lambda^n z) \to (z, z)$ as $n \to \infty$ and $\pi(z, z) = \frac{1}{2}$, we shall show that (24) contradicts (v).

[29] Recall that $0 \in \text{int}(\Sigma)$.

Given Lemma A and the fact that $x < y$, (24) can only hold if $z > y > x$ (as in Figure 2) or $z < x < y$. Consider the former case, the latter being handled similarly. Then

$$\pi(x, z) < \pi(x, y) < \frac{1}{2} = \pi(z, z)$$

and (v) ensures that $\pi(x, y) = \pi(x\mu z, z)$ for some some $\mu \in (0, 1)$. Hence, by (iv),

$$\pi(x, y) < \pi(x\eta z, z) < \pi(x\eta z, y\eta z)$$

for all $\eta < \mu$. This contradicts (24).

We have therefore shown that π is constant on $\{(x, y) \in \Sigma \times \Sigma \mid x - y = k\}$ for any fixed $k \in \mathbb{R}$. From this fact and Lemma A we deduce (5). This completes the proof of Theorem 0.

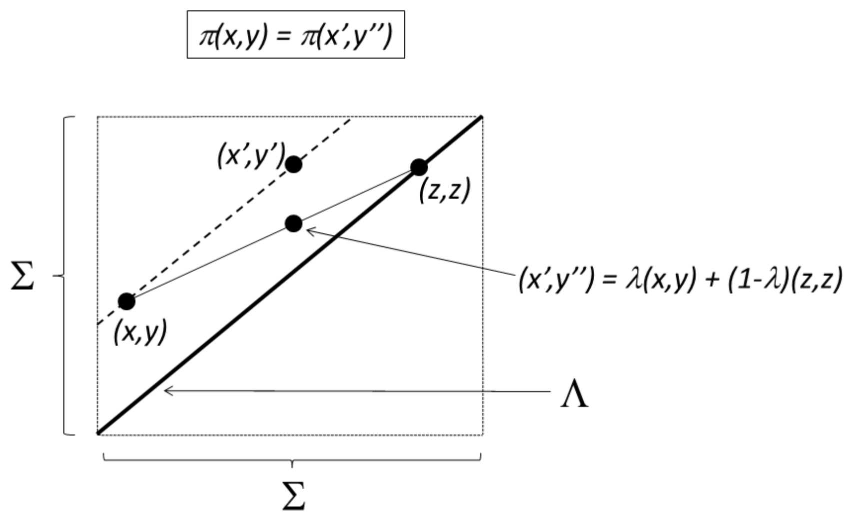

Figure 2

A.2 Proof of Corollary 0

Let $\Sigma = u(M) = u(A)$. Since u is M-linear, Σ is an interval in $\mathbb{R}$. The result is trivial if Σ is a singleton, so we assume otherwise henceforth. It is without loss of generality to suppose that 0 is contained in the interior of Σ – if not, apply a suitable affine transformation to u.

The weak substitutability condition (4), together with (1), imply that

$$u(a) = u(b) \quad \Leftrightarrow \quad P(a, b) = \frac{1}{2} \quad \Rightarrow \quad P(a, c) = P(b, c) \quad \Leftrightarrow \quad P(c, a) = P(c, b)$$

for any $a, b, c \in A$. It follows that there exists a function $\pi : \Sigma \times \Sigma \to [0, 1]$ such that $P(a, b) = \pi(u(a), u(b))$ for any $a, b \in A$. It suffices to show that π satisfies properties (i)–(v) of Theorem 0.

Property (i) is immediate from the definition of u. Property (ii) follows from (1). Strong M-Independence and the M-linearity of u imply

$$\pi(x, y) = \pi(x', y') \quad \Rightarrow \quad \pi(x\lambda z, y\lambda z) = \pi(x'\lambda z, y'\lambda z)$$

for any $x, y, x', y' \in \Sigma$ and any $z \in u(M)$. Since $u(M) = u(A) = \Sigma$, property (iii) holds. Property (iv) follows from SST via the weak substitutability property (4).

Finally, to establish (v) note that π inherits solvability: for any $x, y, z \in \Sigma$ and any $q \in [0, 1]$, if $\pi(z, x) \geq q \geq \pi(z, y)$ then $\pi(z, w) = q$ for some $w \in \Sigma$. From solvability (of π) and weak monotonicity – property (iv) – we easily deduce that π is continuous in its second argument. Continuity in its first argument therefore follows from (ii).

B Strengthening Theorem D

The following theorem strengthens Theorem D.

Theorem D* If A is the unit simplex in $\mathbb{R}^n$ and P satisfies SST, D2, D3 and Strong Independence, then P has a strong expected utility representation.

Proof. The proof is the same as for Proposition 1, except that we can no longer appeal to Lemma 1. Instead, we show that $\succsim^P$ possesses a linear representation by an alternative route.

As per the proof of Proposition 1, we may assume that $\succsim^P$ is a weak order satisfying

$$a \succsim^P b \quad \Rightarrow \quad a\lambda c \succsim^P b\lambda c \tag{25}$$

for all $a, b, c \in A$ and all $\lambda \in (0, 1)$. Since the result is obvious if $\succsim^P$ is trivial (i.e., $a \sim^P b$ for all $a, b \in A$), let us assume otherwise. From D3, we also know that $\succsim^P$ satisfies the following Archimedean property: for any $a, b, c \in A$

$$a \succ^P b \succ^P c \quad \Rightarrow \quad a\lambda c \succ^P b \succ^P a\mu c \tag{26}$$

for some $\lambda, \mu \in (0, 1)$. It therefore suffices (Fishburn, 1982, Chapter 2) to establish that

$$a \succ^P b \quad \Rightarrow \quad a\lambda c \succ^P b\lambda c \tag{27}$$

for all $a, b, c \in A$ and all $\lambda \in (0, 1)$.

Lemma B Condition (27) holds on the interior of A (that is, for any $a, b, c \in A \cap \mathbb{R}^n_{++}$).

Proof. Suppose $a, b, c \in A \cap \mathbb{R}^n_{++}$ with $a \succ^P b$ and $a\lambda c \sim^P b\lambda c$.[30] That is,

$$P(a,b) > \frac{1}{2} \quad \text{and} \quad P(a\lambda c, b\lambda c) = \frac{1}{2} \tag{28}$$

We claim that

$$P(a,b) \geq P(d,e) \geq \frac{1}{2} \quad \Rightarrow \quad P(d\lambda c, e\lambda c) = \frac{1}{2} \tag{29}$$

for any $d, e \in A$. To see why, observe that Strong Independence and

$$P(d,e) \geq \frac{1}{2} = P(c,c)$$

give

$$P(d\lambda c, e\lambda c) \geq \frac{1}{2}.$$

The reverse inequality follows by applying Strong Independence to $P(a,b) \geq P(d,e)$ and using (28).

We next show that $d = a\mu b$ and $e = b$ satisfy the antecedent in (29) for any $\mu \in [0,1]$. The cases $\mu \in \{0,1\}$ are trivial so we focus on $\mu \in (0,1)$.

Since

$$P(a,b) > \frac{1}{2} = P(b,b) = P(a,a)$$

we have

$$P(a\mu b, b) \geq \frac{1}{2} \tag{30}$$

and

$$P(a, a\mu b) = P(a, b(1-\mu)a) \geq \frac{1}{2}$$

by Strong Independence. From the latter inequality and weak substitutability (SST) we have

$$P(a,b) \geq P(a\mu b, b) \tag{31}$$

Thus, from (29)–(31):

$$P((a\mu b)\lambda c, b\lambda c) = \frac{1}{2} \tag{32}$$

for any $\mu \in [0,1]$.

[30] Note that $b\lambda c \succ^P a\lambda c$ is ruled out by (25) and $a \succ^P b$.

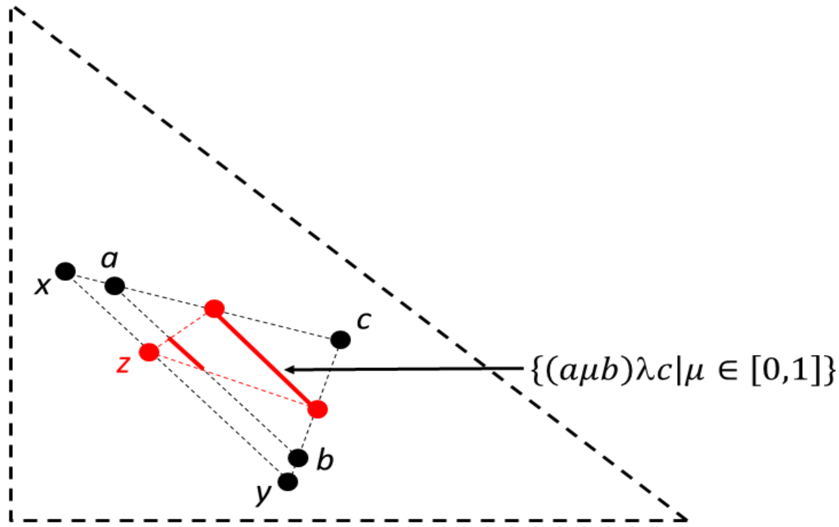

Figure 3

From (32) we obtain a linear segment of strictly positive length,[31] parallel to the line joining a and b, which forms part of an indifference curve for $\succsim^P$. Since a and b are in the interior of the simplex, we can use the existence of this linear segment, together with transitivity of $\succsim$ and the independence property (25), to deduce that the line segment joining a and b must also be part of an indifference curve. Figure 3 illustrates the required construction (for the case $n = 3$). By moving point z along the segment from x to y we sweep out an indifference curve joining a to b.[32] This is the desired contradiction, since $a \succ^P b$. $\square$

Since $A \cap \mathbb{R}^n_{++}$ is a mixture set (under the usual mixing operation for $\mathbb{R}^n$), it follows that $\succsim^P$ possesses a linear representation on $A \cap \mathbb{R}^n_{++}$. Let u be such a representation.

Observe that D3 is not required for the proof of Lemma B – Strong Independence does all the work. We now use D3 to extend the linear representation to the boundary of the simplex.

[31] Recall that $a \succ^P b$, so $a \neq b$.

[32] Let $\hat{a} = a\lambda c$ and $\hat{b} = b\lambda c$. The points x and y are chosen such that $a = x\gamma\hat{a}$ and $b = y\gamma\hat{b}$ for some $\gamma \in (0,1)$. The independence property (25) implies that $u\left(\left(\hat{a}\mu\hat{b}\right)\gamma\left(x\eta y\right)\right)$ is constant in $\mu \in [0,1]$ for any $\eta \in [0,1]$.

First, u has a unique linear extension to A. We use u to denote the extended function, as no confusion should arise. Let a be a boundary point of A. There are two possibilities: either (i) $u(a) = u(b)$ for some $b \in A \cap \mathbb{R}^n_{++}$ or else (ii) the face of the simplex containing a is part of a contour of u.

For case (i) we show that $a \sim^P b$. (It follows that $a \sim^P b$ for any boundary point a and any interior point b on the same utility contour as a.) If, for example, $a \succ^P b$ then we can find some $c \in A \cap \mathbb{R}^n_{++}$ with $u(c) < u(b) = u(a)$. Hence $a \succ^P b \succ^P c$, but $u(a\lambda c) < u(b)$ for all $\lambda \in (0,1)$ by the linearity of u. Since $a\lambda c \in A \cap \mathbb{R}^n_{++}$ for any $\lambda \in (0,1)$, we have $b \succ^P a\lambda c$ for all $\lambda \in (0,1)$, which contradicts the Archimedean property (26). Assuming $b \succ^P a$ leads similarly to contradiction.

Now consider case (ii). Thus, u is constant on the face of the simplex containing a. Suppose, in particular, that $u(a) > u(b)$ for all $b \in A \cap \mathbb{R}^n_{++}$. The alternative scenario, in which $u(a) < u(b)$ for all $b \in A \cap \mathbb{R}^n_{++}$, may be handled similarly. We will show that $a \sim^P b$ for any b on the same face as a, and $a \succ^P b$ for any $b \in A$ not on the same face as a. Combining with case (i), this shows that $\succsim^P$ is represented by u on all of A.

Suppose $b \in A$ with $u(a) = u(b)$, so b lies on the same face of the simplex as a. If $a \succ^P b$ then $a \succ^P b \succ^P c$ for any $c \in A \cap \mathbb{R}^n_{++}$. Since $b \succ^P a\lambda c \in A \cap \mathbb{R}^n_{++}$ for any $\lambda \in (0,1)$, we deduce a contradiction to (26). Assuming $b \succ^P a$ leads similarly to contradiction.

Finally, let $b \in A$ with $u(a) > u(b)$. Assume, contrary to what we seek to show, that $b \succsim^P a$. Since $u(a) > u(b)$ we may choose some $c, d \in A \cap \mathbb{R}^n_{++}$ with

$$u(a) > u(d) > u(c) > u(b).$$

Therefore, $d \succ^P c \succ^P b \succsim^P a$ and

$$u(d\lambda a) = \lambda u(d) + (1-\lambda) u(a) > u(d) > u(c)$$

It follows that $d\lambda a \succ^P c$ for all $\lambda \in (0,1)$. This contradicts the Archimedean property (26).

This completes the proof of Theorem D*. □

We conjecture, but have not been able to prove, that Theorem D* could be strengthened further, by dropping D3. This would yield a result that implies both Proposition 1 and Theorem D*.