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Abstract

The classical Luce model (Luce, 1959) assumes *positivity* of random choice: each available alternative is chosen with strictly positive probability. The model is characterised by Luce's *choice axiom*. Ahumada and Ülkü (2018) and (independently) Echenique and Saito (2019) define the *general Luce model (GLM)*, which relaxes the positivity assumption, and show that it is characterised by a *cyclical independence (CI)* axiom. Cerreia-Vioglio *et al.* (2021) subsequently proved that the choice axiom characterises an important special case of the GLM in which a *rational* choice function (i.e., one that may be rationalised by a weak order) first selects the acceptable alternatives from the given menu, with any residual indifference resolved randomly in Luce fashion. The choice axiom is thus revealed as a fundamental "canon of probabilistic rationality". This result assumes that choice behaviour is specified for all non-empty, finite menus that can be constructed from a given universe, X, of alternatives. We relax this assumption by allowing choice behaviour to be specified for an arbitrary collection of non-empty, finite menus. In this context, we show that the Cerreia-Vioglio *et al.* (2021) result obtains when the choice axiom is replaced with a mild strengthening of CI. The latter condition implies the choice axiom, thus providing a "stricter canon".

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1 Introduction

A *random choice function* specifies the probability, $p(x, E)$, that alternative x is chosen when the decision-maker is confronted with menu E. Alternatives come from some universal domain, X, and menus are non-empty, *finite* subsets of X. The classical Luce model (Luce, 1959) generates choice probabilities from a utility function, $v : X \to \mathbb{R}_{++}$. The probability of choosing alternative $x \in E$ from menu $E \subseteq X$ is equal to the utility of x as a proportion of the total utility of alternatives in E:

$$p(x, E) \;=\; \frac{v(x)}{\sum_{y \in E} v(y)} \qquad (*)$$

This model embodies an assumption of *positivity*: any element of any menu is chosen with strictly positive probability. Positivity is therefore a maintained assumption in the classical literature on the Luce model. Results that "characterise" this model – that specify necessary and sufficient conditions on choice probabilities for the existence of a Luce model representation – typically make additional assumptions about X and the set of menus for which choice probabilities are defined. Luce (1959) assumes a finite X. He characterises the model when the menu set is *comprehensive* – containing all non-empty, finite subsets of X – and also for the *binary* case – when the menu set consists of all binary subsets. In the former case, the model is characterised by the *choice axiom* or, equivalently, *independence of irrelevant alternatives (IIA)*; in the latter, by the *product rule.*

Recently, a number of researchers have defined and characterised generalisations of the Luce model that do not require the positivity assumption.[1] We focus on two such contributions.

The first is due to Ahumada and Ülkü (2018) and (independently) Echenique and Saito (2019). These authors introduce the *general Luce model (GLM).*[2] The GLM requires the existence of a function, $v : X \to \mathbb{R}_{++}$ such that *non-zero* choice probabilities are determined by v in Luce fashion, but with the denominator of $(*)$ modified to be the total utility in the support of $p(\cdot, E)$, rather than the entire menu. Thus, if $p(x, E) > 0$ then

$$p(x, E) \;=\; \frac{v(x)}{\sum_{y \in \Gamma_p(E)} v(y)}$$

[1]A non-exhaustive list would include Ahumada and Ülkü (2018), Cerreia-Vioglio *et al.* (2021), Doğan and Yıldız (2021), Echenique and Saito (2019) and Horan (2021). The latter includes a very thorough and insightful survey of this literature.

[2]We adopt the terminology of Echenique and Saito (2019). Ahumada and Ülkü (2018) refer to this model as the *Luce rule with limited consideration.*

where $\Gamma_p(E)$ denotes the support of $p(\cdot, E)$.

When X is finite and the menu set comprehensive, Ahumada and Ülkü (2018) and Echenique and Saito (2019) show that a generalisation of IIA called *cyclical independence* characterises the GLM.[3] In fact, as we have shown elsewhere (Rodrigues-Neto, Ryan and Taylor, 2024 [RRT24]), this characterisation holds for arbitrary X and for any collection of non-empty, finite menus.

The second contribution is by Cerreia-Vioglio *et al.* (2021). They allow X to be arbitrary but assume a comprehensive menu set. They think of Γ_p as a choice function that selects the "acceptable" elements of the menu. They show that the choice axiom characterises the set of general Luce models for which the mapping Γ_p can be rationalised by a weak order on X. We call this model a *rationalisable GLM.* The choice axiom therefore emerges as a fundamental "canon of probabilistic rationality".

The comprehensive menus assumption is not redundant to Cerreia-Vioglio *et al.*'s (2021) result. We show (Theorem 5) that a mild strengthening of the cyclical independence condition characterises the rationalisable GLM when choice probabilities are defined for an arbitrary collection of non-empty, finite menus. We call this strengthened condition *strong cyclical independence.* It is equivalent to CI when choice probabilities satisfy positivity. It also implies the choice axiom, thus yielding a "stricter canon".

In summary: *whether or not positivity of choice probabilities is assumed, any model that is characterised by the choice axiom when the menu set is comprehensive, is characterised by strong cyclical independence when the menu set is an arbitrary collection of non-empty, finite subsets of* X .

2 The Luce model and its generalisations

2.1 The classical Luce model

Let X be a non-empty set, interpreted as the universal domain of alternatives. Let $\mathcal{M}$ be a non-empty collection of non-empty, *finite* subsets of X. An element of $\mathcal{M}$ is a "menu" from which a single alternative must be chosen – abstention is not allowed. If $\mathcal{M}$ contains all the non-empty, finite subsets of X then we say that the menu set is *comprehensive.*

A *random choice function (RCF)* describes the stochastic choice behaviour of some individual. An RCF specifies a probability function on each menu in $\mathcal{M}$; it is a mapping $p : X \times \mathcal{M} \to [0, 1]$ satisfying $\sum_{x \in A} p(x, A) = 1$ for any $A \in \mathcal{M}$ and $p(x, A) = 0$ for any

[3] We again adopt the terminology of Echenique and Saito (2019). The cyclical independence condition is not named by Ahumada and Ülkü (2018), but appears as their Axiom 1.

$A \in \mathcal{M}$ and any $x \in X \diagdown A$. We interpret $p(x, A)$ as the probability that the individual chooses x when confronted with menu A. For notational convenience, define

$$p(B, A) = \sum_{x \in B} p(x, A)$$

for any $B \subseteq X$ and $A \in \mathcal{M}$.

It is without loss of generality to assume that $\mathcal{M}$ includes all singletons, so we make this assumption throughout. The definition of a random choice function fixes its value on any singleton menu.

If p is an RCF we define $\Gamma_p : \mathcal{M} \to 2^X$ to be the support function for p:

$$\Gamma_p(A) \;=\; \{x \in A \mid p(x, A) > 0\}$$

for each $A \in \mathcal{M}$. Note that Γ_p satisfies the properties of a choice function: $\emptyset \neq \Gamma_p(A) \subseteq A$ for each $A \in \mathcal{M}$. We say that Γ_p is *rationalisable* if there exists a weak order $\succsim \subseteq X \times X$ such that

$$\Gamma_p(A) \;=\; \{x \in A \mid x \succsim y \;\; \text{for all } y \in A\}$$

for each $A \in \mathcal{M}$.

Next, we recall some properties of RCFs and a classical result:[4]

Definition 1 *An RCF, $p : X \times \mathcal{M} \to [0, 1]$, satisfies* ***positivity*** *if $p(x, A) > 0$ when $x \in A \in \mathcal{M}$.*

Definition 2 *An RCF, $p : X \times \mathcal{M} \to [0, 1]$, satisfies the* ***choice axiom (CA)*** *if*

$$p(x, A) \;=\; p(x, B)\, p(B, A)$$

whenever $A, B \in \mathcal{M}$ and $x \in B \subseteq A$.

Definition 3 *An RCF, $p : X \times \mathcal{M} \to [0, 1]$, satisfies* ***independence of irrelevant alternatives (IIA)*** *if*

$$p(x, A)\, p(y, B) = p(x, B)\, p(y, A)$$

whenever $A, B \in \mathcal{M}$ and $\{x, y\} \subseteq A \cap B$.

[4] The IIA condition is usually expressed in ratio form. The version below is equivalent under the usual Luce assumption of positivity.

Definition 4 *An RCF, $p : X \times \mathcal{M} \to [0,1]$, has a* ***Luce model (LM)*** *if there exists some (utility) function $v : X \to \mathbb{R}_{++}$ such that*

$$p(x, A) = \frac{v(x)}{\sum_{y \in A} v(y)}$$

whenever $x \in A \in \mathcal{M}$. We say that v is a Luce model for p.

Theorem 1 (Luce, 1959) *Let X be finite and let $p : X \times \mathcal{M} \to [0,1]$ be an RCF. Suppose $\mathcal{M}$ is comprehensive and p satisfies positivity. Under these assumptions, the following are equivalent:*

(i) p satisfies CA.

(ii) p satisfies IIA.

(iii) p has a Luce model

The assumption of finite X is actually redundant to this classical result. Standard arguments show that (i) is equivalent to (ii) even without the assumption of finite X. Theorem 1 in RRT24 establishes that (ii) is also equivalent to (iii) for arbitrary X.

2.2 The general Luce model

Positivity is obviously a necessary condition for an RCF to possess a Luce model. Ahumada and Ülkü (2018) and Echenique and Saito (2019) relax the positivity assumption and consider the following generalisation of the LM:

Definition 5 *Let $p : X \times \mathcal{M} \to [0,1]$ be an RCF. Then p has a* ***general Luce model (GLM)*** *if there exists a (utility) function $v : X \to \mathbb{R}_{++}$ such that*

$$p(x, A) = \frac{v(x)}{\sum_{y \in \Gamma_p(A)} v(y)}$$

whenever $A \in \mathcal{M}$ and $x \in \Gamma_p(A)$. We say that v is a GLM for p.

Note that any LM is a GLM; and if v is a GLM for p and p satisfies positivity, then v is a LM for p.

One may think of $\Gamma_p(A)$ as the "acceptable" choices from menu A, with v used to randomly resolve "indifference" in Luce fashion. Alternatively – the interpretation favoured by Ahumada and Ülkü (2018) – we may think of $\Gamma_p(A)$ as a *consideration set*, to which the decision-maker restricts attention for the purpose of choice. In this paper, we remain agnostic as to interpretation of the model; our concern is with its "empirical signature" for an arbitrary menu set. However, for the special case of the *rationalisable GLM*, which is our main focus here, the former interpretation is arguably more natural.

Ahumada and Ülkü (2018, Theorem 1) and Echenique and Saito (2019, Theorem 1) show that when X is finite and $\mathcal{M}$ is comprehensive, an RCF has a general Luce model iff it satisfies a condition known as *cyclical independence*. Theorem 3 in RRT24 shows that the assumptions of finite X and comprehensive $\mathcal{M}$ are redundant to this result.

To define cyclical independence we require some additional notation and terminology, mostly adapted from Echenique and Saito (2019). A *connected sequence* is any sequence of the form $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $m \in \{1, 2, ...\}$ and $\{x_i, x_{i+1}\} \subseteq E_i \in \mathcal{M}$ for each i (and repetition allowed). A *cycle* of *length* m is a connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ with $x_1 = x_{m+1}$. Given an RCF, $p : X \times \mathcal{M} \to [0, 1]$, the connected sequence $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is *positive* if $p(x_i, E_i)p(x_{i+1}, E_i) > 0$ for each i. Of course, all connected sequences are positive when p satisfies positivity. A positive connected sequence that is also a cycle is called a *positive cycle*.

Definition 6 *The RCF, $p : X \times \mathcal{M} \to [0, 1]$, satisfies* ***cyclical independence (CI)*** *if*

$$\prod_{i=1}^{m} p\left(x_i, E_i\right) \;=\; \prod_{i=1}^{m} p\left(x_{i+1}, E_i\right) \qquad (\clubsuit)$$

for any positive cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$.

It is evident that CI implies IIA when positivity is assumed. Indeed, CI is equivalent to IIA when $\mathcal{M}$ is comprehensive and p satisfies positivity: see the proof of Theorem 1 in RRT24. When $\mathcal{M}$ comprises all (singleton and) binary menus, the *product rule* requires ($\clubsuit$) to hold for cycles with $m \leq 3$. Horan (2021) therefore calls CI the *strong product rule.*

Theorem 2 (RRT24) *Suppose $p : X \times \mathcal{M} \to [0, 1]$ is an RCF. Then p possesses a GLM if and only if it satisfies CI.*

As noted above, Ahumada and Ülkü (2018, Theorem 1) and Echenique and Saito (2019, Theorem 1) proved the special case of this result for finite X and comprehensive $\mathcal{M}$. The proof of Theorem 2 requires only minor modifications to their arguments.

Since the Luce model and general Luce model are equivalent when p satisfies positivity we have:

Corollary 1 *Suppose $p : X \times \mathcal{M} \to [0,1]$ is an RCF satisfying positivity. Then p possesses a Luce model if and only if it satisfies CI.*

These results characterise the Luce model and general Luce model, respectively, for arbitrary X and arbitrary (finite) menu sets. The CI condition is common, with positivity added in the Luce model characterisation.

2.3 The rationalisable GLM

An important special case of the GLM is obtained if Γ_p is rationalisable. This special case has been studied by various authors under a range of names: Ahumada and Ülkü (2018) call it a *Luce rule with rationalisable consideration* while Doğan and Yıldız (2021) call it a *preference oriented Luce rule.* Within the present paper we shall refer to it as a *rationalisable GLM.*

Definition 7 *An RCF, $p : X \times \mathcal{M} \to [0,1]$, has a **rationalisable GLM** if it has a GLM and Γ_p is rationalisable.*

Cerreia-Vioglio *et al.* (2021) provide an important characterisation of the rationalisable GLM:[5]

Theorem 3 (Cerreia-Vioglio et al., 2021; Theorem 2) *Suppose $\mathcal{M}$ is comprehensive and $p : X \times \mathcal{M} \to [0,1]$ is an RCF. Then p has a rationalisable GLM iff it satisfies CA.*

The comprehensive menus assumption is not redundant in Theorem 3. The choice axiom loses too much bite if the menu set is not sufficiently rich. For example, if $\mathcal{M}$ contains no (non-singleton) menu that is properly contained in another menu, then CA has no bite whatsoever.

3 The main result

Our main result (Theorem 5) triangulates between Theorem 2 and Theorem 3. We seek a necessary and sufficient condition for the existence of a rationalisable GLM for an arbitrary menu set, without assuming positivity.

[5] Alternative characterisations for the special case of finite X are obtained by Ahumada and Ülkü (2018, Corollary 2), Horan (2012, Theorem 4* of the online Appendix) and, in strikingly novel fashion, by Doğan, S. and Yıldız, K. (2021, Theorem 1).

Theorem 5 builds on work by two of the co-authors (Rodrigues-Neto (2009), Fiorini and Rodrigues-Neto (2017) and Taylor (2019a,b)) on the closely related *common prior* problem of Harsanyi (1967-1968). As the Luce formulation ($*$) makes clear, and as has often been noted (including by Luce himself: see pp.10-11 of Luce (1959)), constructing a Luce model when X is finite is analogous to constructing a prior that rationalises a set of posteriors: the conditioning events are the menus and choice probabilities are reinterpreted as posterior probabilities. In particular, when X is finite we may normalise any Luce model so that v sums to 1 over its domain. Under this reinterpretation of p, the choice axiom is the defining characteristic of a *conditional probability space* – a concept that goes back (at least) to Rényi (1955) and Császár (1955).[6]

In the common prior context, the assumption of a "comprehensive" set of conditioning events is far from natural. It is therefore unsurprising that conditions analogous to CI also play a role in the analysis of the common prior problem. We elaborate further upon these connections in RRT24.

Consider the following strengthening of the CI condition:

Definition 8 *An RCF, $p : X \times \mathcal{M} \to [0,1]$, satisfies* ***strong cyclical independence (SCI)*** *if ($\clubsuit$) holds for any cycle $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$.*

Strong cyclical independence requires that the *cycle equation* ($\clubsuit$) holds for every cycle, not just the positive ones. There is obviously no difference between CI and SCI when p satisfies positivity. It turns out that SCI is precisely the condition we seek.

Before stating our main result, we make some preliminary observations on rationalisability of Γ_p. When $\mathcal{M}$ is comprehensive, this is equivalent to Γ_p satisfying the weak axiom of revealed preference (WARP) – a property that is underwritten by the choice axiom (Cerreia-Vioglio *et al.*, 2021). For an arbitrary collection of (finite) menus, a stronger condition is required. To state it, we first define a pair of revealed preference relations: for any $x, y \in X$

$$x \succsim^p y \quad \Leftrightarrow \quad x \in \Gamma_p(E) \text{ and } \{x, y\} \subseteq E \text{ for some } E \in \mathcal{M}$$

$$x \succ^p y \quad \Leftrightarrow \quad \{x, y\} \cap \Gamma_p(E) = \{x\} \text{ and } \{x, y\} \subseteq E \text{ for some } E \in \mathcal{M}$$

Note that $\succ^p \subseteq \succsim^p$ but $\succ^p$ need not be (asymmetric or) the asymmetric part of $\succsim^p$.

Definition 9 (Richter, 1966) *The choice function $\Gamma_p : \mathcal{M} \to 2^X$ satisfies* **congruence** *if there does not exist any sequence $\{x_i\}_{i=1}^m \subseteq X$ with $x_i \succsim^p x_{i+1}$ for each $i \in \{1, ..., m-1\}$ and $x_m \succ^p x_1$.*

[6] This observation is also commonplace in the literature on Luce models.

Restricting the congruence condition to sequences with $m = 2$ gives WARP. The following is well-known (e.g., Chambers and Echenique, 2016, Theorem 2.6):[7]

Theorem 4 *The choice function $\Gamma_p : \mathcal{M} \to 2^X$ is rationalisable iff it satisfies congruence.*

We can now state our main result:

Theorem 5 *Suppose $p : X \times \mathcal{M} \to [0,1]$ is an RCF. Then p possesses a rationalisable GLM iff it satisfies strong cyclical independence. In particular, if p satisfies SCI then Γ_p satisfies congruence.*

Proof. We start with the "if" part of the claim. Suppose p satisfies SCI. Let

$$\mathcal{M}^* = \{\Gamma_p(A) \mid A \in \mathcal{M}\}$$

be the collection of support sets. Since $\mathcal{M}$ contains all singletons, so does $\mathcal{M}^*$ and it follows that $X = \bigcup \mathcal{M}^*$. Define p^* to be the RCF on $X \times \mathcal{M}^*$ which is obtained by setting $p^*(x, \Gamma_p(A)) = p(x, A)$ for each $x \in X$ and each $A \in \mathcal{M}$. We show that p^* is well-defined by strong cyclical independence: if $\Gamma_p(A) = \Gamma_p(B)$ then $p(\cdot, A) \equiv p(\cdot, B)$. To see why, suppose $\Gamma_p(A) = \Gamma_p(B) = E$ and $\{x, y\} \subseteq E$. Then

$$\{(x, y, A), (y, x, B)\}$$

is a positive cycle so

$$p(y, A)\, p(x, B) = p(y, B)\, p(x, A) \quad \Leftrightarrow \quad \frac{p(y, A)}{p(x, A)} = \frac{p(y, B)}{p(x, B)}.$$

Since $p(\cdot, A)$.and $p(\cdot, B)$ both sum to 1 on E, and E is finite, it follows that $p(\cdot, A)$.and $p(\cdot, B)$ coincide on E (hence on X).

Now observe that p^* satisfies positivity and inherits (strong) cyclical independence from p. Hence, by Corollary 1 there is a Luce model $v : X \to \mathbb{R}_{++}$ for p^*. It follows that if $x \in \Gamma_p(A)$ then

$$p(x, A) = p^*(x, \Gamma_p(A)) = \frac{v(x)}{\sum_{y \in \Gamma_p(A)} v(y)}.$$

It remains to show that Γ_p satisfies congruence. Suppose, by way of contradiction, that there exists a sequence $\{x_i\}_{i=1}^m \subseteq X$ with $x_i \succsim^p x_{i+1}$ for each $i \in \{1, ..., m-1\}$ and

[7] When $\succ^p$ is the asymmetric part of $\succsim^p$, congruence is *Suzumura consistency.* This is a necessary and sufficient condition for $\succsim^p$ to have a weak order extension (Suzumura, 1976) – an important generalisation of Szpilrajn's Extension Theorem. The SCI condition ensures that $\succ^p$ is the asymmetric part of $\succsim^p$, as one may easily verify.

$x_m \succ^p x_1$. Then there exists $\{E_i\}_{i=1}^m \subseteq \mathcal{M}$ such that $\{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ is a cycle (hence $x_{m+1} = x_1$), with $x_i \in \Gamma_p(E_i)$ for each $i \in \{1, ..., m\}$ and $x_{m+1} \notin \Gamma_p(E_m)$. It follows that we have a violation of SCI, since

$$\prod_{i=1}^{m} p\left(x_i, E_i\right) \; > \; 0$$

while

$$\prod_{i=1}^{m} p\left(x_{i+1}, E_i\right) = 0.$$

This proves the "if" part of Theorem 5.

Next, we prove the "only if" part. Suppose v is a GLM for p and Γ_p is rationalisable. Let $\mathcal{I} = \{S_k\}_{k \in \mathcal{K}}$ be the family of indifference classes for the weak order that rationalises Γ_p. It follows that $\mathcal{P}$ is a partition of X, and for any $A \in \mathcal{M}$ there exists some $S \in \mathcal{I}$ such that $\Gamma_p(A) \subseteq S$ and $(A \setminus \Gamma_p(A)) \cap S = \emptyset$.

Let $\mathcal{C} = \{(x_i, x_{i+1}, E_i)\}_{i=1}^m$ be a connected sequence with $x_1 = x_{m+1}$ (i.e., a cycle). We show that (♣) is satisfied by considering three exhaustive cases. First, if the sequence is positive, then:

$$\prod_{i=1}^{m} \frac{p\left(x_i, E_i\right)}{p\left(x_{i+1}, E_i\right)} \; = \; \frac{v(x_1)}{v(x_2)} \cdots \frac{v(x_{m-1})}{v(x_m)} \frac{v(x_m)}{v(x_1)} \; = \; 1$$

so (♣) is satisfied. Second, if there exists $i \in \{1, ..., m\}$ with $\{x_i, x_{i+1}\} \cap \Gamma_p(E_i) = \emptyset$ then both sides of (♣) are zero. Finally, we have the case in which there exists $i \in \{1, ..., m\}$ with $\emptyset \neq \{x_i, x_{i+1}\} \cap \Gamma_p(E_i) \neq \{x_i, x_{i+1}\}$. In this case there must exist $S, S' \in \mathcal{I}$ with $S \neq S'$ such that $x_i \in S$ and $x_{i+1} \in S'$. Since the cycle starts and finishes in the same element of $\mathcal{P}$, it must move "up" and "down" the indifference curve hierarchy (relative to the underlying weak order) along the sequence. Similarly for the reverse sequence. Hence, both sides of (♣) must be zero. Thus, in all three cases, (♣) holds.

This completes the proof of Theorem 5. □

Given Theorem 3 it follows that CA is equivalent to SCI when $\mathcal{M}$ is comprehensive. In general, SCI is stronger – a "stricter canon".[8]

Lemma 1 *Suppose $p : X \times \mathcal{M} \to [0, 1]$ is an RCF that satisfies strong cyclical independence. The p satisfies the choice axiom.*

[8] The following is implied by the (iii)⇒(ii) part of Lemma 6 in Cerreia-Vioglio *et al.* (2021). Since they do not prove this particular implication directly, we include a proof here for completeness; it essentially rehearses the well-known argument that IIA implies the choice axiom (minus the redundant assumption of positivity).

Proof. Suppose $A, B \in \mathcal{M}$ with $A \subseteq B$ and $x \in A$. If $\{x\} = A$ the result is trivial, so suppose $|A| \geq 2$. Let $y \in A \diagdown \{x\}$. By considering the cycle $\{(x, y, A), (y, x, B)\}$ we have

$$p(x, A) p(y, B) = p(x, B) p(y, A)$$

Thus:

$$\begin{aligned} \sum_{y \in A \diagdown \{x\}} p(x, A) p(y, B) &= \sum_{y \in A \diagdown \{x\}} p(x, B) p(y, A) \\ \Leftrightarrow \quad p(x, A) p(A \diagdown \{x\}, B) &= p(x, B) [1 - p(x, A)] \\ \Leftrightarrow \quad p(x, B) &= p(x, A) p(A, B) \end{aligned}$$

□

4 Concluding remarks

Cerreia-Vioglio *et al.* (2021) showed that the choice axiom characterises the rationalisable GLM when the menu set is comprehensive. Our main result establishes that strong cyclical independence characterises the same model for an arbitrary menu set. The SCI condition embodies both cyclical independence, which characterises the GLM, and the congruence condition on the support function, Γ_p.

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