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Abstract

This paper studies the *essential elements* (Puppe, 1996) associated with binary relations over opportunity sets. We restrict attention to binary relations which are reflexive and transitive (pre-orders) and which further satisfy a monotonicity and desirability condition. These are called *opportunity relations* (ORs). Our main results axiomatically characterise two important classes of ORs: those for which any opportunity set lies in the same indifference class as its set of essential elements – the *essential* ORs; and those whose essential element operator is the extreme point operator for some abstract convex geometry (Edelman and Jamison, 1985) – the *convex* ORs. Our characterisation of convex ORs generalises the analysis in Klemisch-Ahlert (1993), who restricts attention to a particular subclass of ACGs known as convex shellings. We present an example which suggests that this latter class is restrictive – there are ACGs which are not convex shellings but which are associated with plausible ORs.

Keywords: Opportunity set, freedom, essential alternative, essential element, abstract convex geometry.

JEL Classification: D60, D63.

1 Introduction

In this paper we consider pre-orders (reflexive and transitive binary relations, denoted $\succsim$) on the subsets of a non-empty, finite set X. Subsets of X are *opportunity sets,* or *menus*, from which one alternative (a *meal*) may – though not must – be chosen. If $A, B \in 2^X$, then $A \succsim B$ means that the opportunity (or freedom of choice) represented by A is weakly preferred to that represented by B. Given this interpretation, it is natural to restrict attention to pre-orders that are monotone with respect to set inclusion ($B \subseteq A$ implies $A \succsim B$). We also assume that any non-empty opportunity set is strictly preferred to the empty set. A pre-order satisfying these two additional requirements will be called an *opportunity relation (OR)*. Opportunity relations are the basic objects of study in the paper.

We will not be prescriptive about the basis on which opportunities (or freedoms) are ranked. Various logics are discussed in the literature, and ably surveyed elsewhere.[1] Rankings may follow the instrumentalist logic of indirect utility, or they might take account of factors other than the decision-maker's preferences (if such exist) over X. If opportunity sets are *given* rather than *chosen*, then the decision-maker may feel happier about a social situation that allocates to him a wide freedom of choice even if he would reject most of the offered alternatives. He might also prefer to be offered additional alternatives – besides those he is inclined to choose – if he feels that respected others might find them attractive.

Within the class of opportunity relations, our interest is in those for which the value of opportunity inheres (for whatever reason) in *specific elements* of the opportunity set, in the sense that the full value of opportunity is carried by this subset of elements and removal of any one will materially diminish the value of opportunity. Of course, we allow that such elements may have value even if they would not be chosen. We also allow that the value of an element may be context-dependent. In particular, our analysis closely follows the spirit of Puppe (1996) and Puppe and Xu (2010), and at somewhat farther remove, that of van Hees (2010).

Puppe (1996) introduced the notion of the *essential elements* of an opportunity set. Given $A \subseteq X$ and $x \in A$, we say that x is essential to A if $A \succ A \setminus \{x\}$, where $\succ$ is the asymmetric part of $\succsim$. We use $e(A)$ to denote the (possibly empty) set of essential elements of A.[2] Two of Puppe's (1996) proposed axiomatic restrictions on $\succsim$ are *preference for Freedom of choice (F)*, which requires $e(A) \neq \emptyset$ for all non-empty $A \subseteq X$, and

[1]For example, by Barberà, Bossert and Pattanaik (2004) and Dowding and van Hees (2009).

[2]Throughout the paper, the specific pre-order $\succsim$ giving rise to e is suppressed in the notation but should be obvious from the context.

Independence of Non-Essential alternatives (INE), which requires $A \sim e(A)$ for any $A \subseteq X$ (where $\sim$ is the symmetric part of $\succsim$).[3] Axiom F says that every non-empty set has at least one essential alternative, while INE says that the value of the opportunities represented by A resides exclusively within the essential elements of A.

In the present paper we derive these two important properties of an opportunity relation from a more basic axiom (Theorem 1). We call an OR satisfying F and INE an *essential opportunity relation (EOR).*

For an EOR there is an obvious sense in which the elements of $e(A)$ are the "extreme points" of A: removing an essential element from A strictly diminishes opportunity, and the set $A \setminus e(A)$ is "opportunity-spanned" by $e(A)$ by virtue of the INE property. This suggests a natural connection with the ideas of Klemisch-Ahlert (1993). Suppose that the elements of X are points in $\mathbb{R}^n$. Each point may, for example, correspond to a single commodity described in terms of n measurable characteristics. Klemisch-Ahlert considers a scenario in which adding a new commodity x to an existing opportunity set A will *strictly* improve the value of opportunity (i.e., $A \cup \{x\} \succ A$) provided x is not contained in the convex hull of A. Conversely, if x *is* contained in the convex hull of A, then $A \cup \{x\} \sim A$. In other words, when x is in the convex hull of A, then x is "opportunity-spanned" by the existing alternatives and therefore adds no value, while additions that strictly expand the convex hull of the set are strictly valuable.[4] This restriction on $\succsim$ is called *convex hull monotonicity.* It is straightforward to verify that an OR satisfying convex hull monotonicity must be an EOR.

In the language of *abstract convex geometries* – which are briefly reviewed in Appendix B – convex hull monotonicity implies that $e(A)$ consists of the *extreme points* of A according to the *convex shelling geometry* on $X \subseteq \mathbb{R}^n$ (Edelman and Jamison, 1985). However, there are many abstract convex geometries on X which are not convex shellings. More importantly, there exist EORs for which the essential element mapping e is the extreme point operator for some abstract convex geometry, but not for any convex shelling geometry. Example 4 in Section 5 illustrates this possibility. In this sense, the convex hull monotonicity requirement is unduly restrictive: by limiting attention to convex shellings, we exclude EORs which are nevertheless consistent with the principle that the essential points of a set should be its extreme points relative to some underlying convex structure.

[3] The INE acronym comes from Puppe and Xu (2010). Puppe (1996) calls this property Axiom I, which has less mnemonic value.

[4] Klemisch-Ahlert (1993, p.196) provides three justifications for assuming that the value of a set is the same as that of its convex hull.

In addition to this practical limitation, there is also a theoretical concern with the Klemisch-Ahlert framework. The specific notion of convexity that underpins the convex hull monotonicity condition on $\succsim$ depends on the particular structure of X. This places significant limitations on the application of the model. The set X is obtained by mapping observable alternatives to an abstract space, and the form of this mapping determines which elements of a given set are extreme, and hence which elements are essential according to $\succsim$. It follows that this mapping is observable only if essentiality can be *objectively* determined *a priori*. This will not usually be the case.

In many situations of interest, essentiality will be subjective – it will vary from one decision-maker to the next. The mapping – let's call it f – from observable alternatives to $\mathbb{R}^n$ will be specific to the individual and not known to an outside observer. This mapping determines the subjective notion of convexity that characterises the elements of opportunity sets which the individual regards as essential. In this case, for the model to be useful, it should endogenously (i.e., axiomatically) restrict the *pairs* $(f, \succsim)$ that characterise the desired class of choice behaviours (or preferences over observable opportunity sets).

The present paper is in this spirit. We allow the individual's notion of essentiality – and hence her perception of convexity – to be subjective. We also permit subjective convexity to be described by any abstract convex geometry (ACG), not just by convex shellings. This allows us to dispense with the mapping f and to maintain the conventional assumption that X is objective data, directly observable to a third party. All individual-specific data are confined to the opportunity relation $\succsim$. For a given X, we obtain conditions on an OR which are equivalent to requiring convex hull monotonicity with respect to *some* ACG on X (Theorem 3). The individual's subjective ACG is revealed through her preferences. We call such an OR a *convex opportunity relation (COR)*. It can be verified that a convex opportunity relation is an EOR (Theorem 3).

Abstract convex geometries are a sub-class of *closure spaces* (see Appendix A), for which the notion of extreme points is also well defined. The convex hull of a set its "closure" with respect to forming convex combinations. Closure spaces provide an algebraic abstraction of the general notion of a closure operation. In the context of a closure space (or closure operator), an element x is an extreme point of A if x is not contained in the closure of $A \setminus \{x\}$. The following is therefore a natural generalisation of the notion of convex hull monotonicity: an OR satisfies *closure monotonicity (CM)* if there is some closure operator (on the subsets of X) such that $A \cup \{x\} \sim A$ iff x is in the closure of A. Necessary and sufficient conditions for an opportunity relation to exhibit CM are given in Section 4. We call such an OR a *closed opportunity relation (ClOR)*.

It turns out that a ClOR need not be an EOR. Moreover, the set of ClOR's which *are* EOR's coincides exactly with the set of COR's (Theorem 3).[5]

The next section introduces the notion of an opportunity relation. The essential opportunity relations are characterised in Section 3. It is convenient to analyse closed opportunity relations before introducing convex opportunity relations (i.e., to progress from less restrictive to more restrictive classes of ORs), so the former are studied in Section 4 and the latter in Section 5. Section 6 concludes. The Appendices contain background material on closure spaces (Appendix A) and abstract convex geometries (Appendix B).

2 Opportunity Relations

First, some notation. Throughout the paper, X will denote a non-empty, finite set and $\succsim$ will denote a pre-order (reflexive and transitive binary relation) on 2^X. We define $\succ$, $\sim$, $\precsim$ and $\prec$ from $\succsim$ in the usual way. We also omit brackets around singleton subsets of X whenever convenient. Finally, we use $\subseteq$ and $\subset$ to denote subsets and proper subsets respectively.

Given that $\succsim$ is not required to be complete, the restriction imposed by transitivity is mild. If $\succsim$ reflects the rankings that would be elicited by direct interrogation of the individual (rather than imputed from choice behaviour),[6] then it is reasonable to suppose that most individuals would abstain rather than knowingly express rankings which violate transitivity. Such intransitivities are unlikely to be found, under close scrutiny, to be compatible with fully determinate preferences.

We shall be exclusively concerned with pre-orders that satisfy two further properties:[7]

Definition 1. *An **opportunity relation (OR)** is a pre-order $\succsim \subseteq 2^X \times 2^X$ satisfying the following desirability (D) and monotonicity (M) conditions:*

$$A \succ \emptyset \text{ for all non-empty } A \subseteq X \tag{D}$$

$$\text{If } \emptyset \neq B \subseteq A \text{ then } A \succsim B \tag{M}$$

[5] These results mirror familiar properties of closure operators: see Theorem B.2 in Appendix B.

[6] In particular, $\succsim$ need not reflect actual or hypothetical choice behaviour. The individual need not anticipate facing a *choice of* opportunity sets (as opposed to a *choice from* an opportunity set). The binary relation $\succsim$ may instead reflect his preferences over the opportunities with which the world chooses to present him – preferences over what he might be offered, rather than inclinations to choose. This distinction is potentially important for analysing conceptions of freedom.

[7] Since M implies reflexivity we could replace "pre-order" in Definition 1 with "transitive binary relation".

Monotonicity is self-explanatory. If the terms "opportunity" and "freedom" have any ordinal significance at all, surely monotonicity must lie at the heart of it? Puppe and Xu (2010, p.671) remark that M "seems to be an uncontroversial condition and we expect any sensible freedom-ranking should satisfy this condition". The desirability assumption D is typically assumed when the binary relation is defined over all subsets of X, rather than just the non-empty subsets.[8]

While M and D are relatively uncontroversial assumptions in this literature, they are not without substance, and since they are the foundation for all that follows, we briefly rehearse the standard objections and present our defences against them.

The obvious objection – to both M and D – is that X might contain noxious alternatives whose presence degrades an opportunity set. We are not persuaded by this objection for the following reasons.[9]

We wish to understand "opportunity" or "freedom" as notions that delimit what the individual *can* do. Any compulsion should be determined by what is *excluded* from an opportunity set, not what is included. Objections to M or D rely on the confounding effect of implicit compulsions that are assumed to accompany the presentation of an opportunity set. These elements of compulsion muddy the waters, obscuring our view of what "opportunity" or "freedom" entails in its purest sense.

Consider monotonicity. If we add a noxious alternative to a *non-empty* opportunity set, this will degrade the set only if its inclusion somehow compels the decision-maker to contemplate the noxious alternative more vividly than she otherwise might. We assume otherwise. In other words, we assume that the elements of X have all been fully contemplated by the decision-maker before any opportunity set is presented to her. She fully understands the world in which she lives, including its darker elements, and making a conscious choice to avoid unpleasant alternatives imposes no higher psychic cost than being compelled to avoid them. With this interpretation, M is innocuous even if X contains noxious elements.

The desirability assumption is questionable if X contains noxious elements and we assume that choice is "forced", that an alternative *must* be chosen from the opportunity set with which the decision-maker is presented. If so, then presenting the decision-maker with a singleton opportunity set means *imposing* the sole alternative upon her. To avoid the implied compulsion, we maintain the assumption – or rather, the interpretation –

[8] See, for example, Puppe (1996, p.178) and Puppe and Xu (2010, p.671). A notable exception is van Hees (2010).

[9] None of the following reasons is original to the present author, of course.

that *abstention from choice* is always an option; the only option if the opportunity set is empty. Opportunity can only compel by restricting choice, not by imposing it.

Our interpretation therefore requires that clear meaning can be attached to the notion of not choosing, but this does not seem unduly restrictive. Given that abstention is allowed, there is minimal loss of generality in further assuming that X contains only elements that are individually desirable: for any $x \in X$, the decision-maker would strictly prefer to choose than not if her opportunity set were $\{x\}$. We shall make this assumption throughout. Together with M, it implies D.[10]

The rest of the paper characterises various classes of ORs.

As a prelude, it will be useful to define a pair of operators associated with an opportunity relation.

Let $\succsim$ be an OR and let $e : 2^X \to 2^X$ be defined from $\succsim$ as follows: for any $A \subseteq X$,

$$e(A) = \{x \in A \mid A \succ A \diagdown x\}$$

The members of $e(A)$ are called the *essential elements* of A (Puppe, 1996).[11] Removing an essential element reduces the value of the opportunity represented by the set. The following lemma gives an equivalent definition of e.

Lemma 1. *If $\succsim$ is an OR then*

$$e(A) = \bigcap \{B \subseteq A \mid A \sim B\}$$

for any $A \subseteq X$.

Proof. Given M, $A \sim A \diagdown x$ iff $x \notin e(A)$, so

$$\bigcap \{B \subseteq A \mid A \sim B\} \subseteq e(A).$$

To show the reverse inclusion, suppose $x \in e(A)$ and $B \subseteq A \sim B$. We must prove that $x \in B$. If $x \notin B$ then

$$A \sim B \subseteq A \diagdown x.$$

[10] In other words, nothing would be lost by replacing D with the *No Dummy* condition of Danilov, Koshevoy and Savaglio (2015): $\{x\} \succ \emptyset$ for every $x \in X$.

[11] Nehring and Puppe (1999) say that x is "essential at $A \diagdown x$" if $x \in e(A)$. We follow Puppe's (1996) terminology in the present paper. The related notion of an *eligible element* was introduced by van Hees (2010). However, his eligible element mapping $e : 2^X \to 2^X$ is treated as exogenous data, logically separate from the pre-order $\succsim$, though the axioms in van Hees (2010) restrict the relationship between the two objects.

M and transitivity therefore give $A \sim A \setminus x$, which contradicts $x \in e(A)$. $\square$

The second operator is a natural "dual" to e. Given $\succsim$, we define $\sigma : 2^X \to 2^X$ as follows:[12] for any $A \subseteq X$,

$$\begin{aligned} \sigma(A) &= \{x \in X \mid A \sim A \cup x\} \\ &= A \cup \{x \in X \setminus A \mid x \notin e(A \cup x)\} \end{aligned} \tag{1}$$

The set $\sigma(A)$ augments A with all of the elements which, individually, add no value. The following result, which mirrors Lemma 1, gives an equivalent definition of σ.

Lemma 2. *If $\succsim$ is an OR then*

$$\sigma(A) = \bigcup \{B \subseteq X \mid A \subseteq B \text{ and } A \sim B\}$$

for any $A \subseteq X$.

Proof. Since $A \sim A \cup x$ for every $x \in \sigma(A)$, it is obvious that

$$\sigma(A) \subseteq \bigcup \{B \subseteq X \mid A \subseteq B \text{ and } A \sim B\}.$$

Conversely, suppose $A \subseteq B \sim A$ and $z \in B$. Then

$$A \subseteq A \cup z \subseteq B \sim A.$$

Using M and transitivity we deduce $A \sim A \cup z$. That is, $z \in \sigma(A)$. $\square$

3 Essential Opportunity Relations

The elements of $e(A)$ are *individually essential* to the opportunity represented by A. However, they may not be *collectively sufficient*. It is possible that $A \succ e(A)$.

Example 1. *Suppose $X = \{a, b, c\}$ and $\succsim$ is the weak order (i.e., complete pre-order) satisfying*

$$\emptyset \prec a \sim b \sim c \sim \{b, c\} \prec \{a, b\} \sim \{a, c\} \sim X.$$

This is an OR but $e(X) = \{a\} \prec X$.

[12] As for e, we rely on context to determine the OR from which σ is derived.

If $A \sim e(A)$ for all $A \subseteq X$, then $\succsim$ is said to satisfy the *Independence of Non-Essential alternatives (INE)* property (Puppe, 1996). We call an opportunity relation that satisfies INE an *essential opportunity relation (EOR).* For an EOR, the essential elements of A carry the full value of the opportunity represented by A.

The following lemma gives some other useful properties of EORs.

Lemma 3. *Let $\succsim$ be an EOR with essential element mapping e. Then the following hold for any $A, B \in 2^X$:*

(i) If $A \neq \emptyset$ then $e(A) \neq \emptyset$.[13]

(ii) If $B \subseteq A$ then $A \sim B$ iff $e(A) = e(B)$.

(iii) If $e(A) \subseteq B \subseteq A$ then $e(B) = e(A)$.

Proof. To show (i), suppose $A \neq \emptyset$. If $e(A) = \emptyset$ then INE implies $A \sim \emptyset$ which contradicts desirability (D).

Next, consider (ii). If $e(A) = e(B)$ then $A \sim B$ follows by INE and transitivity. Conversely, suppose $B \subseteq A$ and $e(A) \neq e(B)$. It suffices to prove that there exists some $x \in e(A) \setminus B$: from this it follows that $A \succ A \setminus x$ and $B \subseteq A \setminus x$, so $A \succ B$ by M and transitivity. Contrary to what we need to show, suppose

$$e(A) \subseteq B \tag{2}$$

Since $B \subseteq A$, INE, M and transitivity give

$$A \sim B \sim e(A) \sim e(B).$$

From (2) and $e(B) \subseteq B \subseteq A$, we may now deduce $e(A) = e(B)$ by two applications of Lemma 1. This is the desired contradiction.

Finally, consider (iii). If $e(A) \subseteq B \subseteq A$ then INE and M imply $A \sim B$ and the result follows by (ii). $\square$

For an EOR, every non-empty opportunity set contains an essential element (property (i)); removing non-essential elements does not alter the set of essential elements (property (iii));[14] while adding new elements leads to a material improvement in opportunity if and only if it changes the set of essential elements (property (ii)).[15]

[13] This is Axiom F in Puppe (1996).

[14] Property (iii) is a strengthening of the well-known Aizerman (or Outcast) condition on choice functions, which requires $e(B) \subseteq e(A)$ whenever $e(A) \subseteq B \subseteq A$ (Moulin, 1985).

[15] Property (ii) sharpens the "main content" of Puppe (1996, Proposition 1), and also Fact 7.1 in Nehring and Puppe (1999).

INE is imposed as an axiom in Puppe (1996) and Puppe and Xu (2010), but is easily deduced from more elementary properties. Consider the following property, which appears (unnamed) in Bossert, Ryan and Slinko (2009; henceforth BRS):[16]

Definition 2. *An OR $\succsim$ satisfies* ***Collective Contraction Non-essentiality (CCN)*** *if, for all $A \subseteq X$ and all $x, y \in A$,*

$$A \sim A \diagdown x \sim A \diagdown y \quad \Rightarrow \quad A \sim A \diagdown \{x, y\} .$$

Theorem 1. *Let $\succsim$ be an OR. Then $\succsim$ is an EOR iff $\succsim$ satisfies CCN.*

Proof. To show the "if" part, suppose CCN holds and $A = e(A) \cup \{x_1, ..., x_n\}$. Thus $A \sim A \diagdown x_i$ for each $i \in \{1, ..., n\}$. We show that $A \sim A \diagdown B$ for any $B \subseteq \{x_1, ..., x_n\}$ by induction on $|B|$. If $|B| = 1$ this follows by assumption. Let $k \in \{1, ..., n-1\}$ and suppose it is true for $|B| \in \{1, ..., k\}$. Let $B \subseteq \{x_1, ..., x_n\}$ with $|B| = k+1$. It is without loss of generality (WLOG) to assume that $B = \{x_1, ..., x_{k+1}\}$. By transitivity and the inductive hypothesis

$$A \sim A \diagdown \{x_1, ..., x_{k-1}\} \sim A \diagdown \{x_1, ..., x_k\} \sim A \diagdown \{x_1, ..., x_{k-1}, x_{k+1}\}$$

Hence CCN implies

$$A \sim A \diagdown \{x_1, ..., x_{k-1}\} \sim A \diagdown \{x_1, ..., x_{k+1}\} = A \diagdown B.$$

Conversely, suppose CCN does not hold. Then there exists $A \subseteq X$ and $x, y \in A$ such that $A \sim A \diagdown x \sim A \diagdown y$ but $A \succ A \diagdown \{x, y\}$. It follows that $e(A) \subseteq A \diagdown \{x, y\}$ so M implies

$$A \succ A \diagdown \{x, y\} \succsim e(A)$$

Hence $A \succ e(A)$ by transitivity. $\square$

[16] The same property also appears, in a slightly weaker version, in Nehring and Puppe (1999), and in heavily disguised form in Nehring and Puppe (1998). Nehring and Puppe's (1999) weaker version of CCN is called the *Irrelevance of Inessential Elements (IIE)* property. It allows CCN to be violated if $x \sim y \sim \{x, y\}$ or if $A \diagdown \{x, y\} = \emptyset$. Our Example 1 satisfies IIE. Note also that if $\succsim$ is an OR satisfying CCN then $x \sim y \sim \{x, y\}$ implies $x = y$ (otherwise CCN and D are in contradiction). Nehring and Puppe's (1998) *Strict Properness* is essentially equivalent to CCN, though applied to so-called *weak extended partial orders (WEPOs).* Rather than binary relations on opportunity sets (subsets of $2^X \times 2^X$), Nehring and Puppe (1998) work with *extended binary relations on* X, which are subsets of $2^X \times X$. The WEPOs are a particular class of extended binary relations. We may transform a binary relation $\succsim \subseteq 2^X \times 2^X$ into an extended binary relation $Q \subseteq 2^X \times X$ (and *vice versa*) by specifying that $(A, x) \in Q$ iff $A \succsim A \cup x$. Under this transformation, $\succsim$ satisfies CCN iff Q satisfies Strict Properness.

If $\succsim$ is an EOR, then Lemmas 2 and 3 imply that

$$\sigma(A) = \bigcup \{B \subseteq X \mid A \subseteq B \text{ and } e(A) = e(B)\} \tag{3}$$

for any $A \subseteq X$. It is tempting to interpret $\sigma(A)$ as the "opportunity span" (or "opportunity closure") of A, and the elements of $e(A)$ as the critical (or "extreme") points that support this opportunity span. However, as we show in the next section, not all EORs are able to bear this interpretation. The ones that are will be characterised in Section 5.

4 Closed opportunity relations

The notion of a *closure space* (Appendix A) provides an abstract algebraic characterisation of spanning (or closure) operations. Every closure space has an associated *extreme point operator*. The extreme points of a set are the elements whose individual removal would strictly diminish the span (or closure) of the set. Formal definitions are given in Appendix A.

An OR whose essential element operator is the extreme point operator for some closure space will be called a *closed opportunity relation (ClOR)*. Of course, it would be equally natural to say that an OR is "closed" if the associated mapping (1) is the closure operator for some closure space (Appendix A). Fortunately, no ambiguity arises.

Lemma 4. *Let $\succsim$ be an OR. If σ is the closure operator for some closure space on X then e is the associated extreme point operator for that closure space. Likewise, if e is the extreme point operator for some closure space on X, then σ is the associated closure operator.*

Proof. Suppose σ is a closure operator and $x \in e(A)$. That is, $A \succ A \setminus x$ and hence $x \notin \sigma(A \setminus x)$. Since σ is monotone with respect to set inclusion (see property (CC2) in Appendix A) and $x \in \sigma(A)$ it follows that $\sigma(A \setminus x) \subset \sigma(A)$. That is, x is an extreme point of A. Conversely, suppose x is an extreme point of A, so $\sigma(A \setminus x) \subset \sigma(A)$. If $x \in \sigma(A \setminus x)$ then $A \subseteq \sigma(A \setminus x)$ and hence

$$\sigma(A) \subseteq \sigma(\sigma(A \setminus x)) = \sigma(A \setminus x)$$

by the monotonicity and idempotency of σ (properties (CC2) and (CC3) in Appendix A). This contradicts $\sigma(A \setminus x) \subset \sigma(A)$, so we must have $x \notin \sigma(A \setminus x)$. It follows that $A \succ A \setminus x$, and therefore $x \in e(A)$. This proves the first claim. The second follows directly from (1) above and (15) in Appendix A. $\square$

The following example shows that not every EOR is closed.

Example 2. *Suppose $X = \{a, b, c\}$ and $\succsim$ is the following weak order:*

$$\emptyset \prec b \sim c \prec a \sim \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$

This is an OR and satisfies CCN. In fact, it is easily verified that

$$A \sim \sigma(A) \sim e(A)$$

for all $A \subseteq X$. However, $\sigma(a) = \{a, b\}$ while $\sigma(\{a, c\}) = \{a, c\}$, so σ violates the monotonicity property of a closure operator (Property CC2 in Appendix A) which requires that $\sigma(A) \subseteq \sigma(B)$ whenever $A \subseteq B$.

We next introduce a condition that is necessary and sufficient for an OR to be closed (see Theorem 2).

Definition 3. *An OR $\succsim$ satisfies **Expansion Monotonicity (EM)** if, for all $A \subseteq X$ and all $x, y \in X \setminus A$,*

$$A \sim A \cup x \quad \Rightarrow \quad A \cup y \sim A \cup \{x, y\}.$$

Note that the OR in Example 2 violates EM: take $A = \{a\}$, $x = b$ and $y = c$.

Lemma 5 illustrates some useful consequences of EM. To state this result we first introduce the following generalisation of *convex hull monotonicity* (Klemisch-Ahlert, 1993):

Definition 4. *An OR $\succsim$ satisfies **closure monotonicity (CM)** if the following hold for any $A, B \in 2^X$:*

$$\sigma(B) \subseteq \sigma(A) \quad \Rightarrow \quad A \succsim B$$

and

$$\sigma(B) \subset \sigma(A) \quad \Rightarrow \quad A \succ B.$$

Lemma 5. *If $\succsim$ is an OR satisfying EM then:*

(i) $A \sim \sigma(A)$ for any $A \subseteq X$.[17]

[17] Example 2 shows that EM is not necessary for (i). A necessary and sufficient condition is the following: for all $A \subseteq X$ and all $x, y \in X$,

$$A \sim A \cup x \sim A \cup y \quad \Rightarrow \quad A \sim A \cup \{x, y\} \tag{4}$$

To see the sufficiency of (4), note that the proof of Lemma 5(i) does not use the full strength of EM – only (4) is required. To see the necessity, suppose there exists $A \subseteq X$ and $x, y \in X$ such that $A \sim A \cup x \sim A \cup y$ but $A \cup \{x, y\} \succ A$. Then $A \cup \{x, y\} \subseteq \sigma(A)$ so M and transitivity give $\sigma(A) \succ A$.

(ii) $\succsim$ *satisfies CM.*

Proof. Consider (i). Let $\sigma(A) = A \cup \{x_1, ..., x_n\}$. Thus $A \sim A \cup x_i$ for each $i \in \{1, ..., n\}$. We show that $A \sim A \cup B$ for any $B \subseteq \{x_1, ..., x_n\}$ by induction on $|B|$. If $|B| = 1$ this follows by assumption. Let $k \in \{1, ..., n-1\}$ and suppose it is true for $|B| \in \{1, ..., k\}$. Let $B \subseteq \{x_1, ..., x_n\}$ with $|B| = k+1$. It is WLOG to assume that $B = \{x_1, ..., x_{k+1}\}$. By transitivity and the inductive hypothesis

$$A \sim A \cup \{x_1, ..., x_{k-1}\} \sim A \cup \{x_1, ..., x_k\} \sim A \cup \{x_1, ..., x_{k-1}, x_{k+1}\}$$

Hence EM implies

$$A \sim A \cup \{x_1, ..., x_{k-1}\} \sim A \cup \{x_1, ..., x_{k+1}\} = A \cup B.$$

Conversely, suppose CCN does not hold. Then there exists $A \subseteq X$ and $x, y \in A$ such that $A \sim A \diagdown x \sim A \diagdown y$ but $A \succ A \diagdown \{x, y\}$. It follows that $e(A) \subseteq A \diagdown \{x, y\}$ so M implies

$$A \succ A \diagdown \{x, y\} \succsim e(A)$$

Hence $A \succ e(A)$ by transitivity.

Next, we show (ii). If $\sigma(B) \subseteq \sigma(A)$ then $\sigma(A) \succsim \sigma(B)$ by M. Applying Lemma 5(i) and transitivity we deduce $A \succsim B$. If $\sigma(B) \subset \sigma(A)$ then $A \succsim B$ as just shown. If $A \sim B$ then (Lemma 5(i) and transitivity) $\sigma(A) \sim B$. Since $B \subseteq \sigma(B) \subset \sigma(A)$ we have $B \subseteq \sigma(A) \sim B$, and therefore $\sigma(A) \subseteq \sigma(B)$ by Lemma 2. This is the desired contradiction. $\square$

The EM property appears (unnamed) in BRS. For binary relations which are transitive and satisfy M (such as ORs), EM is equivalent to a number of variant conditions that have appeared elsewhere in the literature. For example, taking the contrapositive of EM and applying M, we obtain the following property: for all $B \subseteq X$ and all $x, y \in B$ with $x \neq y$

$$B \succ B \diagdown x \quad \Rightarrow \quad B \diagdown y \succ B \diagdown \{x, y\}$$

Given transitivity, this is equivalent to the *Contraction Consistency (CC)* condition of Nehring and Puppe (1999):[18] for all $A, B \in 2^X$ with $A \subseteq B$ and all $x \in A$

$$B \succ B \diagdown x \quad \Rightarrow \quad A \succ A \diagdown x \tag{5}$$

[18] However, Nehring and Puppe (1999) apply CC to binary relations contained within a restricted subset of $2^X \times 2^X$. The Monotonicity condition on WEPOs (Nehring and Puppe, 1998) can also be translated (via the rule of translation noted previously) into the following version of CC: $A \subseteq B$ and $A \succsim A \cup x$ imply $B \succsim B \cup x$.

Given M and transitivity, (5) is equivalent to the *strict contraction monotonicity* condition (Ryan, 2014):[19] for all $A, B, C \in 2^X$ with $C \subseteq A \subseteq B$

$$B \succ B \diagdown C \quad \Rightarrow \quad A \succ A \diagdown C \tag{6}$$

Taking the contrapositive of (6), we obtain property (1.5) from Kreps (1979):[20] for all $A, B, C \in 2^X$ with $A \subseteq B$

$$A \sim B \quad \Rightarrow \quad A \cup C \sim B \cup C \tag{7}$$

Moreover, if we re-express (5) in terms of essential elements, we obtain the Heritage Axiom (also known as the Chernoff Property or Property α), which is familiar from the literature on choice functions (Moulin, 1985): for all $A, B \in 2^X$

$$A \subseteq B \quad \Rightarrow \quad e(B) \cap A \subseteq e(A) \tag{8}$$

It is well known that the Heritage Axiom is necessary for e to be the extreme point operator for a closure space (Ando, 2006). If e is the essential element mapping for an OR, then (8) is also sufficient.

Theorem 2. *Let $\succsim$ be an OR. Then $\succsim$ satisfies EM iff $\succsim$ is closed.*

Proof. Suppose $\succsim$ is an OR that satisfies EM. It is obvious that $\sigma(\emptyset) = \emptyset$ and that $A \subseteq \sigma(A)$ for any $A \subseteq X$. It remains to verify monotonicity and idempotency – properties (CC2) and (CC3) in Appendix A. Let $A \subseteq B$ and let $x \in X$ be such that $A \sim A \cup x$. If $x \in B$ then $B \sim B \cup x$ by reflexivity. If $x \in X \diagdown B$ then $B \sim B \cup x$ by iterative application of EM for each $y \in B \diagdown A$. Hence $\sigma(A) \subseteq \sigma(B)$. This proves that σ is monotone. It follows that $\sigma(A) \subseteq \sigma(\sigma(A))$ for any $A \subseteq X$. To verify idempotency it suffices to show that $\sigma(\sigma(A)) \subseteq \sigma(A)$. Suppose there exists $x \in \sigma(\sigma(A)) \diagdown \sigma(A)$. Then $A \cup x \succ A$. Since $\sigma(A) \cup x \succsim A \cup x$ by M, we have

$$\sigma(A) \cup x \succ A \tag{9}$$

[19] Suppose $B \succ B \diagdown C$ and let $C = \{c_1, ..., c_n\}$. By M and transitivity there exists some $k^* \in \{1, ..., n-1\}$ such that

$$B \diagdown \{c_1, ..., c_{k^*}\} \succ B \diagdown \{c_1, ..., c_{k^*+1}\}.$$

Then, using M, transitivity and (5) we have:

$$A \succsim A \diagdown \{c_1, ..., c_{k^*}\} \succ A \diagdown \{c_1, ..., c_{k^*+1}\} \succsim A \diagdown C$$

and hence $A \succ A \diagdown C$.

[20] See also Nehring and Puppe (1999, footnote 3).

by transitivity. By applying Lemma 5(i) and transitivity to (9), we have $\sigma(A) \cup x \succ \sigma(A)$, which contradicts $x \in \sigma(\sigma(A))$.

Conversely, let $\succsim$ be a ClOR. Suppose $A \sim A \cup x$. Then $x \in \sigma(A) \subseteq \sigma(A \cup y)$, so $A \cup y \sim A \cup \{x, y\}$. $\square$

Theorem 2 shows that the defining characteristic of a closed opportunity relation is the Heritage property: an essential element remains essential if other elements (essential or otherwise) are removed. Furthermore, Lemma 5(ii) says that the Heritage property implies closure monotonicity: if the closure of A (properly) contains the closure of B, then A is (strictly) preferred to B.

It is important to observe that EM does *not* imply CCN: not every ClOR is an EOR.

Example 3. *Let $X = \{x, y\}$ and consider the weak order on 2^X given by*

$$\emptyset \prec x \sim y \sim X.$$

It is trivial to confirm that this is an OR which satisfies EM but violates CCN.

The next section characterises the ClORs which are also EORs.

Before concluding this section, let us observe that a *complete* ClOR has an *expected indirect utility* representation. This follows from Kreps (1979, Theorem 1) and the fact that an OR satisfies EM iff it satisfies (7). In fact, we can say more: if $\succsim$ is a ClOR then there exists a complete ClOR $\succsim'$ that extends $\succsim$ (in the sense that $\succ \subseteq \succ'$ and $\sim \subseteq \sim'$) and has an expected indirect utility representation. It is straightforward to show that any pre-order on 2^X can be extended to a weak order. If the original pre-order is a ClOR then so is any extension, since D, M and EM only restrict its non-extended part.

5 Convex opportunity relations

An *abstract convex geometry (ACG)* is a closure space for which the closure operation is analogous to forming a convex hull (Appendix B). An example of an ACG is a *convex shelling geometry.* This arises if X is a finite set of points in $\mathbb{R}^n$ and the closure of A consists of the points in X contained in the (Euclidean) convex hull of A.

An OR will be called a *convex opportunity relation (COR)* if σ is the closure operator for some ACG. The convex opportunity relations comprise a subset of the closed opportunity relations, so every COR is a ClOR. However, not every closed opportunity relation is convex. In fact, an opportunity relation is convex precisely when it is both closed and essential.

Theorem 3. *Let $\succsim$ be a ClOR. Then $\succsim$ is a COR iff $\succsim$ satisfies CCN. That is, the CORs consist of the ClORs which are also EORs.*

Proof. Since $\succsim$ is a ClOR, σ is a closure operator (Theorem 2) and e its associated extreme point operator (Lemma 4). We must therefore show that σ satisfies the anti-exchange property (Appendix B) iff $\succsim$ satisfies CCN.

Suppose σ satisfies the anti-exchange property and $A \sim A \diagdown x \sim A \diagdown y$, where $\{x, y\} \subseteq A \subseteq X$. Then

$$x \in \sigma [(A \diagdown \{x, y\}) \cup y] \quad \text{and} \quad y \in \sigma [(A \diagdown \{x, y\}) \cup x] \tag{10}$$

Defining $E = \sigma (A \diagdown \{x, y\})$, the monotonicity of σ and (10) imply $x \in \sigma (E \cup y)$ and $y \in \sigma (E \cup x)$. Since σ satisfies the anti-exchange property, we must have $x \in E$ or $y \in E$. Thus, either $A \diagdown \{x, y\} \sim A \diagdown x$ or $A \diagdown \{x, y\} \sim A \diagdown y$, from which we deduce $A \diagdown \{x, y\} \sim A$.

Conversely, suppose $\succsim$ is a ClOR that satisfies CCN. Let E be a closed set and let $x, y \in X \diagdown E$. Then $E \cup \{x, y\} \succ E$ by CM (Lemma 5). Suppose $x \in \sigma (E \cup y)$. We cannot have $y \in \sigma (E \cup x)$ since this would imply

$$E \cup \{x, y\} \sim E \cup y \sim E \cup x$$

from which $E \cup \{x, y\} \sim E$ follows by CCN. $\square$

Thus, when an OR satisfies both CCN and EM, the value of opportunity resides entirely within its essential elements (INE) and there is a subjective ACG on X such that a point is essential to a given set precisely when it is an extreme point with respect to this ACG. In particular, CORs satisfy closure monotonicity (Lemma 5).

Klemisch-Ahlert (1993) studies a sub-class of CORs in which $X \subseteq \mathbb{R}^n$ for some n and the associated ACG is the convex shelling geometry for X. As noted in the Introduction, this implies that the ACG can be determined from X – it is "objective" data – which is a restrictive assumption. Moreoever, there are many ACGs which cannot be realised as convex shelling geometries (Kashiwabara *et al.*, 2005). Theorem 3 therefore generalises Klemisch-Ahlert's analysis. It determines the ACG endogenously, and permits a wider range of convex structures with which to describe the decision-maker's subjective notion of essentiality.

The following example suggests that this generalisation is not without interest.

Example 4. *Let $X = \{a, b, c, d\}$, where the elements of X correspond to the pure strategies of Player 2 (the column player) in the following two-player game:*[21]

	a	b	c	d
α	$-$, 1	$-$, 6	$-$, 3	$-$, 4
β	$-$, 6	$-$, 1	$-$, 3	$-$, 0

*We may therefore identify each $x \in X$ with a vector in $\mathbb{R}^4$. Define $\kappa : 2^X \to 2^X$ as follows: $\kappa(A)$ consists of A together with any elements of $X \diagdown A$ which are **strictly dominated** by some mixture over the pure strategies in A. It can be shown that κ is the closure operator for an ACG over X (Kukushkin, 2004). Now, for each $A \subseteq X$, define $\kappa^*(A)$ to be the convex shell of $\kappa(A) \subseteq \mathbb{R}^4$ in X; that is,*

$$\kappa^*(A) \;=\; \mathrm{co}(\kappa(A)) \cap X$$

where $\mathrm{co}(E)$ denotes the Euclidean convex hull of $E \subseteq \mathbb{R}^4$. It is straightforward (and not too tedious) to verify that $\kappa^* \equiv \kappa$ for this example.[22] *Finally, define a binary relation $\succsim$ as follows: $A \succsim B$ iff $|\kappa^*(A)| \geq |\kappa^*(B)|$, where $|E|$ denotes the cardinality of the set E.*[23]

It is easy to verify that $\succsim$ is an EOR.[24] *Furthermore,*

$$e(A) \;=\; \{x \in A \mid \kappa^*(A \diagdown x) \subset \kappa^*(A)\}$$

for any $A \subseteq X$, so e is the extreme point operator associated with κ^*. It follows that $\succsim$ is a COR and that $\sigma = \kappa^*$ (Lemma 4). In other words, the decision-maker regards a pure strategy $x \in A$ to be essential to A iff it is neither payoff-equivalent to, nor strictly dominated by, any mixture of the other elements of A.

[21]Only Player 2's payoffs are shown; Player 1's payoffs are redundant to the analysis.

[22]In fact, consider *any* finite two player game. Let n be the number of pure strategies available to Player 1 and let $X \subseteq \mathbb{R}^n$ be the set of (Player 2) payoff vectors corresponding to each of Player 2's pure strategies. It is not hard to show that the following defines an anti-exchange closure operator:

$$\kappa^*(A) \;=\; \mathrm{co}\left(A + \mathbb{R}^n_-\right) \cap X \;=\; \left(\mathrm{co}(A) + \mathbb{R}^n_-\right) \cap X.$$

[23]The fact that this decision-maker ranks sets by the cardinality of their *closure* may strike the reader as odd. Why should adding a strictly dominated strategy, for example, increase the value of the strategic opportunity set? While this may seem odd, it is nevertheless consistent with the well-known *decoy (or attraction) effect* (De Clippel and Eliaz, 2012). That said, any monotone pre-order satisfying $A \sim \kappa^*(A)$ for all $A \subseteq X$ and $A \succ B$ for all $A, B \in 2^X$ with $\kappa^*(B) \subset \kappa^*(A)$ will suit our purpose just as well.

[24]In particular, the monotonicity property of a closure operator (CC2) ensures that M is satisfied.

To see that κ^* cannot be the closure operator for any convex shelling geometry, observe that

$$\kappa^* (\{a, b\}) \; = \; X \tag{11}$$

(since the equal mixture of a and b strictly dominates c) and

$$\kappa^* (\{b, d\}) \; = \; \{b, d\} \tag{12}$$

If κ^* were the closure operator for some convex shelling geometry, then the elements of X could be identified with points in $\mathbb{R}^n$ (for some n) such that $\kappa^* (A)$ is the intersection of X with the (Euclidean) convex hull of A. From (11) we would therefore deduce that points c and d lie on the line segment in $\mathbb{R}^n$ joining a to b. Hence, (12) implies that c is between a and d. However, the latter implication is contradicted by the fact that[25]

$$\kappa^* (\{a, d\}) \; = \; \{a, d\} \, .$$

One of the three justifications provided by Klemisch-Ahlert (1993, p.196) for assuming that the value of a set is the same as that of its convex hull, is that the decision-maker is able to choose by explicit randomisation and may regard the associated lottery as value-equivalent to its "expected realization". Example 4 is entirely consistent with this justification, since mixtures are regarded as equivalent to pure strategies that deliver the same expected payoff for each rival strategy. In particular, $\mathrm{co}\,(A) \cap X \subseteq \kappa^* (A)$ for any $A \subseteq X$. But this does not exclude the possibility that the decision-maker might perceive a "coarser" convex structure, such that $\mathrm{co}\,(A) \cap X$ may be strictly contained in the closure of A. As in Example 4, it will not always be possible to describe this coarser structure with a convex shelling geometry.

6 Related literature

Theorem 2 and Theorem 3 connect our work with related results elsewhere in the literature. In this section we briefly clarify these connections. It may be skipped without loss of continuity.

[25] To ensure a payoff greater than 3 when Player 1 chooses β, Player 2 must place probability greater than $\frac{1}{2}$ on a when mixing over a and d. But then the payoff to this mixture will be less than $\frac{5}{2}$ when Player 1 chooses α.

6.1 Theorem 2

Our Theorem 2 is a close relative of Proposition 5 in Danilov, Koshevoy and Savaglio (2015) [henceforth DKS]. As DKS observe, this result has been independently re-discovered in variant forms by several authors, with Kreps (1979, Lemmas 1 and 2) being its first appearance in the economics literature.

DKS characterise what they call the *transitive decent hyper-relations.* In our terminology, a transitive decent hyper-relation $\succsim^*$ is an OR which satisfies the following *Union* property: for any $A, B, C \in 2^X$

$$[C \succsim^* A \text{ and } C \succsim^* B] \quad \Rightarrow \quad C \succsim^* A \cup B$$

Proposition 5 in DKS shows that if $\succsim^*$ is a transitive decent hyper-relation, then the mapping $\mu : 2^X \to 2^X$ defined by

$$\mu(A) = \{x \in X \mid A \succsim^* x\} \tag{13}$$

is a closure operator.

To connect the DKS result with ours, recall that the *dominance (or domination) relation* $\succsim^* \subseteq 2^X \times 2^X$ associated with a given binary relation $\succsim \subseteq 2^X \times 2^X$ is defined thus (Kreps, 1979):

$$A \succsim^* B \quad \Leftrightarrow \quad A \succsim A \cup B$$

Observe that μ is the mapping (1) expressed in terms of the associated dominance relation, since (given M) we have:

$$A \succsim^* x \quad \Leftrightarrow \quad A \succsim A \cup x \quad \Leftrightarrow \quad A \sim A \cup x$$

for any $A \subseteq X$ and any $x \in X$.

The dominance relation associated with an OR will obviously satisfy M (hence reflexivity) and also D, but need not be transitive.[26] If it is transitive, then μ is equivalent to the mapping $\phi : 2^X \to 2^X$ given by

$$\phi(A) = \bigcup \{B \subseteq X \mid A \succsim^* B\}$$

Kreps (1979) studies this mapping and proves that if $\succsim$ is a complete ClOR then its dominance relation $\succsim^*$ is transitive (*ibid.*, Lemma 1) and ϕ is a closure operator (*ibid.*, Lemma 2).[27]

[26] It also satisfies conditions (Cont) and (Ext) of DKS.

[27] More precisely, Kreps studies weak orders on $2^X \setminus \{\emptyset\}$ that satisfy M and (7). The latter, as we have observed, is equivalent to EM (given M and transitivity). If we use D to extend such a weak order to 2^X, we obtain a complete ClOR.

The following result therefore ties together our Theorem 2 with Kreps (1979, Lemmas 1 and 2) and DKS (Proposition 5):

Proposition 1. *Let $\succsim$ be an OR with associated dominance relation $\succsim^*$. The following are equivalent:*

(a) $\succsim$ is a ClOR

(b) $\succsim^*$ is transitive (i.e., an OR).

(c) $\succsim^*$ is transitive and satisfies the Union property.

Proof. Kreps (1979, Lemma 1) implies that (a) implies (b) if $\succsim$ is complete. We can deduce that (a) is equivalent to (b) even without completeness from Puppe (1996, Lemma 1). In particular, any ClOR satisfies Puppe's Axiom F (Theorem 1) and F is not used in Puppe's proof that (b) implies (a) (*ibid.*, p.195).

Since (c) obviously implies (b), it remains to verify that the dominance relation of a ClOR satisfies the Union property. Therefore, let $\succsim$ be a ClOR with dominance relation $\succsim^*$. From the discussion following Theorem 1, if $A \succsim^* B$ and $A \succsim^* C$ then M implies $A \sim A \cup B$ and $A \sim A \cup C$. Hence, from Lemma 3(ii), we deduce $e(A \cup B) = e(A \cup C) = e(A)$. Combining this fact with the Heritage property we have

$$e(A \cup B \cup C) \cap (A \cup B) \subseteq e(A)$$

and

$$e(A \cup B \cup C) \cap (A \cup C) \subseteq e(A)$$

and therefore $e(A \cup B \cup C) \subseteq e(A)$. Applying M and INE gives $A \succsim^* B \cup C$. $\square$

Thus, our Theorem 2 is essentially the result of Kreps (1979) in modified form (and without the unnecessary assumption of completeness of the OR). The result of DKS provides a complementary characterisation of the dominance relations associated with ClOR's.[28] Of course, it is obvious that a ClOR cannot, in general, be recovered from its dominance relation. Nevertheless, the parts of an OR which cannot be recovered from

[28] In a similar vein, Nehring and Puppe (1998, Theorem 4.1) apply Kreps (1979) to characterise the *weak extended partial orders (WEPO's)* associated with ClOR's. Given an OR $\succsim$, its associated WEPO consists of the pairs (A, x) such that $A \succsim^* x$, where $\succsim^*$ is the dominance relation for $\succsim$. A yet-more-distant relative of the same basic result, but expressed in the idiom of modal logic, is presented in Gekker and van Hees (2006).

its dominance relation are inconsequential to establishing whether or not it is a ClOR, as is clear from the fact that EM only restricts an OR on "nested" pairs of sets (i.e., pairs where one set is contained in the other).

6.2 Theorem 3

Combining Theorems 2 and 3 with Lemma 4 and Theorem 1, we obtain:

Corollary 1. *Let $\succsim$ be an OR. Then $\succsim$ satisfies EM and CCN iff σ is an anti-exchange closure operator and $A \sim e(A)$ for all $A \subseteq X$.*

Corollary 2. *Let $\succsim$ be an OR. Then e is the extreme point operator for some ACG iff $\succsim$ satisfies EM and INE.*

Corollary 1 is a variation on the main result in BRS. In the BRS version, $\succsim$ is assumed to be a weak order (i.e., a complete and transitive binary relation) that satisfies D but not necessarily M.

Corollary 2 strengthens Corollary 4.4 in Puppe and Xu (2010) by demonstrating the redundancy of Axiom F. (Recall that an OR satisfies EM iff it satisfies (8), which is Sen's (1971) Property α.) Likewise, Axiom F may be dropped from Puppe and Xu's Proposition 4.7; less obviously, also from Puppe (1996, Proposition 2).[29]

Similarly to Theorem 2, DKS also provide a complementary perspective on Theorem 3. They characterise the dominance relations associated with COR's.[30] Section 6 of DKS considers the sub-class of transitive, decent hyper-relations which also satisfy the following condition:[31] for any $A, B \in 2^X$

$$A \sim^* B \quad \Rightarrow \quad A \cap B \succsim^* A \tag{14}$$

DKS refer to these as *ample hyper-relations.*[32]

Danilov, Koshevoy and Savaglio (2015, Proposition 1 and Theorem 3) show that $\succsim^*$ is an ample hyper-relation iff the mapping $\psi : 2^X \to 2^X$ defined by

$$\psi(A) \;=\; \{x \in A \mid (A \diagdown x, x) \notin \succsim^*\}$$

[29]Properties M and α already imply both F and INE (cf, Puppe, 1996, Lemma 2). Our Proposition 1 shows that F is redundant to Puppe's Lemma 1.

[30]With some effort, one can also perceive Nehring and Puppe's (1998) Theorem 5.2 as an essentially equivalent result for WEPO's.

[31]Condition (14) is equivalent to the *Lattice Equivalence (LE)* condition in DKS, given the Union property and the finiteness of X.

[32]Note that transitivity of $\succsim^*$ actually follows from the other properties of an ample hyper-relation: Danilov, Koshevoy and Savaglio (2015, p.67).

is the extreme point operator for some ACG. Observe that if $\succsim$ is an OR with associated dominance relation $\succsim^*$, then ψ is the essential point operator for $\succsim$: given $x \in A \subseteq X$

$$(A \diagdown x, x) \notin \succsim^* \quad \Leftrightarrow \quad A \succ A \diagdown x \quad \Leftrightarrow \quad x \in e(A)$$

(where we have made use of M). Given our Proposition 1, the following is therefore a corollary of the DKS result, but we provide a direct proof.

Proposition 2. *Let $\succsim$ be a ClOR with dominance relation $\succsim^*$. Then $\succsim$ is a COR iff $\succsim^*$ satisfies (14).*

Proof. Let $\succsim$ be a COR. Then $\succsim$ is essential (Theorems 1 and 3). Suppose $A \sim^* B$. Then $A \sim A \cup B \sim B$ so Lemma 3(ii) implies $e(A \cap B) = e(A)$. Applying INE and transitivity we conclude that $A \cap B \sim A$, from which $A \cap B \succsim^* A$ follows.

Conversely, let $\succsim$ be a ClOR whose dominance relation $\succsim^*$ satisfies (14). We must show that $\succsim$ satisfies CCN (Theorem 3). Let $\{x, y\} \subseteq A \subseteq X$ with $A \sim A \diagdown x \sim A \diagdown y$. Then $A \diagdown x \sim^* A$ and $A \diagdown y \sim^* A$. Since $\succsim^*$ is transitive (Proposition 1), $A \diagdown x \sim^* A \diagdown y$. Hence (14) implies $A \diagdown \{x, y\} \succsim^* A$, which is the desired conclusion. $\square$

7 Concluding remarks

The class of opportunity relations provides a simple and natural environment for exploring the ranking of opportunity sets. Our analysis reveals how two restrictions on OR's – CCN and EM – underpin a range of important principles in the analysis of "freedom rankings", including Puppe's (1996) Axiom F and the Independence of Non-Essential alternatives (INE) property, as well as a generalised version of Klemisch-Ahlert's (1993) convex hull monotonicity. Both of these conditions (EM and CCN) are simple and transparent. They also delineate the opportunity relations whose essential element mappings are the extreme point operators of closure spaces or ACG's. Our results, in this latter respect, complement the analysis of DKS, whose characterisations take the form of restrictions on dominance relations.

According to Theorem 3, if the value of opportunity resides precisely in essential elements – the INE property – and coincides with the "opportunity span" of these essential elements according to some coherent subjective spanning criterion (i.e., some well-defined closure operator), then essential elements behave as extreme points relative to some underlying ACG. Convex structure therefore lies at the heart of such opportunity relations.

This is arguably surprising, but certainty convenient: convex structure brings with it many useful and well-known properties.

While convex shellings provide a rich class of convex structures for discrete environments, they do not exhaust the ACGs. Example 4 shows that ACGs outside the convex shelling class may be necessary to describe ORs that are far from exotic. Restricting attention to convex shellings, as in Klemisch-Ahlert (1993), imposes substantive restrictions.

Finally, as will be clear from our discussion of the Related Literature (Section 6), many of our results are synthetic rather than organic. They could have been derived in large part by suitable translations from existing results for dominance relations. Nevertheless, this would have been a needlessly circuitous route. ORs are straightforward to interpret, and axioms CCN and EM are simple and clear. The connections between CCN, EM and the important notions of essentiality, closedness and convexity of an OR are obscure in the previous literature. We hope that our elucidation of these connections will facilitate further research. Section 6 may also be of independent interest in this regard.

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Appendices

The following Appendices review some basic facts about closure spaces and abstract convex geometries. Further details, including omitted proofs, can be found in Edelman and Jamison (1985) unless otherwise specified.

A Closure spaces

Given a finite set X, a *closure space on* X is a collection $\mathcal{K}$ of subsets of X satisfying, for all $A, B \subseteq X$:

(C0) $\emptyset, X \in \mathcal{K}$,

(C1) If $\{A, B\} \subseteq \mathcal{K}$, then $A \cap B \in \mathcal{K}$.

We take X as given from now on and omit the qualifier "on X" when discussing closure spaces.

A closure space is an abstract generalisation of the notion of closing a set with respect to some underlying operation, such as taking limits or forming linear combinations. The elements of $\mathcal{K}$ are interpreted as the subsets of X which are closed with respect to the underlying operation. They are therefore called the *closed* subsets of X.

Given a closure space $\mathcal{K}$ and a set $A \subseteq X$, we define $\sigma_{\mathcal{K}}(A)$ to be the smallest element of $\mathcal{K}$ containing A. This is well-defined by (C1):

$$\sigma_{\mathcal{K}}(A) \;=\; \bigcap \{B \in \mathcal{K} \mid A \subseteq B\}\,.$$

We say that $\sigma_{\mathcal{K}}(A)$ is the *closure* of A. It is easy to see that $\mathcal{K}$ and $\sigma_{\mathcal{K}}$ contain the same information: given $\sigma_{\mathcal{K}}$ we may recover $\mathcal{K}$ by the rule: $A \in \mathcal{K}$ iff $\sigma_{\mathcal{K}}(A) = A$.

More generally, we call $\sigma : 2^X \to 2^X$ a *closure operator* if it satisfies, for all $A, B \subseteq X$:

(CC0) $\sigma(\emptyset) = \emptyset$,

(CC1) $A \subseteq \sigma(A)$,

(CC2) $A \subseteq B$ implies $\sigma(A) \subseteq \sigma(B)$,

(CC3) $\sigma(\sigma(A)) = \sigma(A)$.

Defining

$$\mathcal{K}_\sigma = \{A \in X \mid \sigma(A) = A\}$$

it follows easily that $\mathcal{K}_\sigma$ is a closure space;[33] and given any closure space $\mathcal{K}$, it is clear that $\sigma_\mathcal{K}$ satisfies (CC0)–(CC3). Indeed:

Theorem A.1. *Given a closure space $\mathcal{K}$, the operator $\sigma_\mathcal{K}$ is a closure operator and $\mathcal{K} = \mathcal{K}_{\sigma_\mathcal{K}}$.*

Proof. We already observed that $\sigma_\mathcal{K}$ is a closure operator. If $A \in \mathcal{K}$, then $\sigma_\mathcal{K}(A) = A$. Conversely, if $\sigma_\mathcal{K}(A) = A$ then A is the intersection of sets in $\mathcal{K}$ and hence $A \in \mathcal{K}$. □

Given a closure operator σ, we may also define the operator $\mathrm{ex}_\sigma : 2^X \to 2^X$ as follows:[34]

$$\mathrm{ex}_\sigma(A) = \{x \in A \mid \sigma(A) \neq \sigma(A \setminus x)\}$$

The elements of $\mathrm{ex}_\sigma(A)$ are the *extreme points* of A.

The closure operator σ can be recovered from ex_σ as follows:[35]

$$\sigma(A) = A \cup \{x \in A^c \mid x \notin \mathrm{ex}_\sigma(A \cup x)\} \tag{15}$$

Thus, $\mathcal{K}$, $\sigma_\mathcal{K}$ and $\mathrm{ex}_\mathcal{K}$ all encode the same information.

[33] To verify (C1), suppose $\sigma(A) = A$ and $\sigma(B) = B$. Two applications of (CC2) gives

$$\sigma(A \cap B) \subseteq \sigma(A) \cap \sigma(B) = A \cap B.$$

Since $A \cap B \subseteq \sigma(A \cap B)$ by (CC1), we are done.

[34] Some authors use the equivalent definition

$$\mathrm{ex}_\sigma(A) = \{x \in A \mid x \notin \sigma(A \setminus \{x\})\}.$$

[35] If $x \in A^c$, then $x \notin \mathrm{ex}_\sigma(A \cup x)$ clearly implies $x \in \sigma(A)$. Conversely, suppose $x \in \sigma(A)$. Then $A \cup x \subseteq \sigma(A)$ by (CC1), and hence, using (CC2) and (CC3):

$$\sigma(A \cup x) \subseteq \sigma(\sigma(A)) = \sigma(A).$$

B Abstract convex geometries

An *abstract convex geometry (ACG)* is a closure space for which the closure operation has the (algebraic) flavour of forming a convex hull. Formally, an ACG is a closure space $\mathcal{K}$ that satisfies:

(C2) If $A \in \mathcal{K} \setminus \{X\}$, then $A \cup x \in \mathcal{K}$ for some $x \in A^c$.

The sense in which (C2) captures the idea of convexity is clarified by the following important result (see, for example, Edelman and Jamison, 1985).

Theorem B.1. *If $\mathcal{K}$ is an ACG, then the associated closure operator $\sigma_{\mathcal{K}}$ satisfies:*

(CC4) For any $A \subseteq X$ with $\sigma(A) = A$ and any distinct $x, y \in A^c$, if $y \in \sigma(A \cup x)$, then $x \notin \sigma(A \cup y)$.

Conversely, if σ is a closure operator satisfying (CC4), then $\mathcal{K}_\sigma$ satisfies (C2).

Condition (CC4) is called the *anti-exchange property.*

Edelman and Jamison (1985) provide a range of other conditions on a closure space that are equivalent to the anti-exchange property of its associated closure operator. For our purposes, the most important of these is the following generalisation of the Minkowski-Krein-Milman property:

Theorem B.2 (Edelman and Jamison, Theorem 2.1)**.** *A closure space $\mathcal{K}$ is an ACG iff $\sigma_{\mathcal{K}}(A) = \sigma_{\mathcal{K}}(\mathrm{ex}_{\mathcal{K}}(A))$ for any $A \subseteq X$.*

Thus, the extreme points of A "carry" the convex hull (closure) of A.

The most familiar examples of ACG's are the so-called *convex shelling geometries.* Suppose X is a finite subset of $\mathbb{R}^n$ (or the image of some one-to-one mapping of the original set into $\mathbb{R}^n$). Let $A \in \mathcal{K}$ iff

$$A = X \cap \mathrm{co}(A)$$

where $\mathrm{co}(A)$ is the usual (Euclidean) convex hull of A in $\mathbb{R}^n$. Then $\mathcal{K}$ is a convex shelling geometry. It is easily verified that a convex shelling is closure space and that it satisfies (C2).